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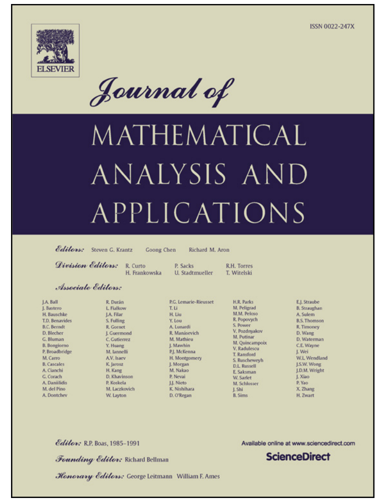

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GAUSSIAN DECAY OF HARMONIC OSCILLATORS AND RELATED MODELS

B. CASSANO AND L. FANELLI

Abstract. We prove that the decay of the eigenfunctions of harmonic oscillators, uniform electric or magnetic fields is not stable under 0-order complex perturbations, even if bounded, of these Hamiltonians, in the sense that we can produce solutions to the evolutionary Schrödinger flows associated to the Hamiltonians, with a stronger Gaussian decay at two distinct times. We then characterize, in a quantitative way, the sharpest possible Gaussian decay of solutions as a function of the oscillation frequency or the strength of the field, depending on the Hamiltonian which is considered. This is connected to the Hardy's Uncertainty Principle for free Schrödinger evolutions.

1. Introduction

Let us consider an electromagnetic Schrödinger Hamiltonian of the form

$$H = -\Delta_A + V(x),$$

where $\Delta_A := (\nabla - iA)^2$ and the potentials A, V are given by

$$A : \mathbb{R}^n \to \mathbb{R}^n, \qquad V : \mathbb{R}^n \to \mathbb{R}.$$

We assume that H can be defined as a self-adjoint operator on a suitable subset $X \subset L^2(\mathbb{R}^n)$, so that the Schrödinger flow e^{itH} is well-defined by functional calculus. Moreover, we assume that H has pure point spectrum, and its eigenvalues form an orthonormal basis of $L^2(\mathbb{R}^n)$. This is a typical situation, if unbounded (at infinity) perturbations are involved, like harmonic oscillators or uniform electric or magnetic fields, as we see in the sequel. In this framework, we have a countable set of standing-waves of the form $e^{itH}\psi_k = e^{i\lambda_k t}\psi_k$, being λ_k an eigenvalue and ψ_k a corresponding eigenfunction of H. The space-decay at infinity of these objects is, independently of time, the one of the eigenfunctions ψ_k, which is, in most cases, exponential. The two most relevant models are the following ones.

Example 1.1 (Quantum harmonic oscillator)**.** Consider the 1D-equation

$$i\partial_t u - \partial_{xx} u + \frac{\omega^2 x^2}{4} u = 0. \tag{1.1}$$

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