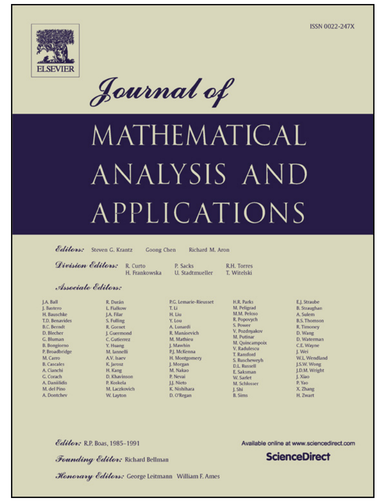

Remarks on sets with small differences and large sums

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REMARKS ON SETS WITH SMALL DIFFERENCES AND LARGE SUMS

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ABSTRACT. We use a surprising example of a subset of group $\mathbb{Z}_{12}$ to construct the set $X \subset \mathbb{R}$ with $int(X - X) = \varnothing$ and $int(X + X) \neq \varnothing$. We also build an ideal $\mathcal{J}$ on a compact abelian topological group G such that $\mathcal{J}$ has the closed $(1, 1)$-Steinhaus property and does not have the closed $(1, -1)$-Steinhaus property.

The classical result of Steinhaus [14] says that each Lebesgue measurable set A of positive measure satisfies the condition $int\,(A - A) \neq \varnothing$. The same condition is true for $A + A$. On the other hand the classic ternary Cantor set has the same properties, although it is a null set. In the eighties, R. Ger asked if there exists a compact set $A \subset \mathbb{R}$ such that $int\,(A - A) \neq \varnothing$ and $int\,(A + A) = \varnothing$. The question was answered by Crnjac, Guljaš and Miller who constructed a compact set S with $S - S = [-1, 1]$ and $\operatorname{int}(S + S) = \varnothing$ ([6]). A similar question was formulated at the conference dedicated to the 20th anniversary of the Chair of Algebra and Topology of Lviv National University that was held in 2001. Taras Banakh asked whether there is a compact subset A of the real line $\mathbb{R}$ with $0 \in int\ (A - A)$ such that the sum $A + A$ (or $A + ... + A$) has an empty interior in $\mathbb{R}$ ([2]).

It is very surprising that the Ger and Banakh problems were fully resolved in the early 70's by British mathematicians Jackson, Williamson, Woodall, Connolly and Haight. They started from an interesting property of subsets of groups $\mathbb{Z}_n$. In the series of papers [10], [5] and [9] they proved that for any positive integer k there exist a number n and a set $E \subset \mathbb{Z}_n$ with $E - E = \mathbb{Z}_n$ and a k-sum $E + ... + E \neq \mathbb{Z}_n$. (We use the notation "+" for the addition in any considered group, because it does not lead to any misunderstanding.) In [7], Connolly and Williamson proved that this property leads to the statement that for any positive integer k there exists a compact subset A of reals such that $A - A$ contains an interval and a k-sum $A + ... + A$ has Lebesgue measure zero.

All mentioned sets were built using auxiliary subsets of groups $\mathbb{Z}_n$. It has appeared that the existence of subsets of $\mathbb{Z}_n$ with some algebraic properties enables constructions:

- compact subsets of $\mathbb{R}$ with similar properies (atractors of affine iterated function systems)
- σ-ideals of subsets of compact groups such that sets not belonging to the σ-ideal (or closed sets which do not belong to the σ-ideal) have "similar properties".

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