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Existence and nonexistence of solutions to elliptic equations involving the Hardy potential

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A B S T R A C T

The purpose of this paper is to study the nonexistence of nonnegative super solutions to the problem

$$(-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u \ge Q u^p \quad \text{in} \quad \mathbb{R}^N \setminus \mathcal{K}, \tag{0.1}$$

where $\alpha \in (0, 1]$, $\mu \in \mathbb{R}$, $p > 0$, $\mathcal{K}$ is a compact set in $\mathbb{R}^N$ with $N \ge 1$ and Q is a potential in $\mathbb{R}^N \setminus \mathcal{K}$ satisfying that $\liminf_{|x|\to+\infty} Q(x)|x|^\gamma > 0$ for some $\gamma < 2\alpha$. When $\alpha = 1$, $(-\Delta)^\alpha$ is the Laplacian operator, and when $\alpha \in (0, 1)$, it is the fractional Laplacian which is a typical nonlocal operator. In this paper, we find the critical exponent $p^* > 1$ depending on α, μ and γ such that problem (0.1) has no nontrivial nonnegative super solutions for $0 < p < p^*$. Furthermore, we also consider the existence and nonexistence of isolated singular solutions to the equation

$$\begin{cases} (-\Delta)^\alpha u + \dfrac{\mu}{|x|^{2\alpha}} u = Q u^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\ \lim\limits_{|x|\to+\infty} u(x) = 0, \end{cases}$$

where $\mu > 0$, $p > 0$ and $Q(x) = (1 + |x|)^{-\gamma}$ with $\gamma \in (0, 2\alpha)$.

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1. Introduction

We are concerned with the nonexistence of nontrivial nonnegative super solutions to the problem

$$(-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u \ge Q u^p \quad \text{in} \quad \mathbb{R}^N \setminus \mathcal{K}, \tag{1.1}$$

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where $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $p > 0$, $\mathcal{K}$ is a compact set in $\mathbb{R}^N$ with $N \geq 1$ and Q is a potential in $\mathbb{R}^N \setminus \mathcal{K}$ satisfying that $\liminf_{|x| \to +\infty} Q(x)|x|^\gamma > 0$ for some $\gamma < 2\alpha$. When $\alpha = 1$, the operator $(-\Delta)^\alpha$ is the Laplacian, and when $\alpha \in (0,1)$, it is the fractional Laplacian defined in the principle value sense as

$$(-\Delta)^\alpha u(x) = c_{N,\alpha} \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(x+z)}{|z|^{N+2\alpha}} dz,$$

where $B_\epsilon(0)$ is the ball centered at the origin with radius ϵ, $c_{N,\alpha}$ is the normalized constant

$$c_{N,\alpha} = 2^{2\alpha} \alpha \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2\alpha}{2})}{\Gamma(1-\alpha)}$$

and Γ is the Gamma function. The fractional Laplacian is a nonlocal operator, so if Lebesgue measure $|\mathcal{K}| \neq 0$, we have to assume moreover that $u \geq 0$ a.e. in $\mathcal{K}$. The semilinear elliptic equations involving the fractional Laplacian and the related Sobolev spaces have been studied extensively, see [1,5,10–12,20,21] and the references therein.

It is known that the fundamental solution and Comparison Principle play an important role in the obtention of the nonexistence of solutions to semilinear elliptic equations. In the Laplacian case, the authors in [2,6] used the fundamental solution of Laplacian and Comparison Principle to obtain the nonexistence of positive solutions to the problem

$$-\Delta u = Q u^p \quad \text{in} \quad \mathbb{R}^N \setminus \mathcal{K}.$$

In the fractional case, i.e. $\alpha \in (0,1)$, [14] shows the nonexistence results of (1.1) when $\mu = 0$, $\mathcal{K} = \emptyset$, $Q = 1$ and $p \leq \frac{N}{N-2\alpha}$, by using the fundamental solution of the fractional Laplacian and Comparison Principle.

To study the nonexistence of nonnegative nontrivial super solutions of (1.1), we first clarify the fundamental solution of $(-\Delta)^\alpha + \frac{\mu}{|x|^{2\alpha}}$ as follows.

Proposition 1.1. *Assume that $\alpha \in (0,1]$ and $N \in \mathbb{N}$.*

(i) When $N > 2\alpha$, let us denote

$$\bar{\tau} = -\frac{N-2\alpha}{2} \quad \textit{and} \quad \mu_0 = \begin{cases} -\frac{(N-2)^2}{4} & \textit{if} \quad \alpha = 1, \\ -2^{2\alpha-1} c_{N,\alpha} \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})} & \textit{if} \quad \alpha \in (0,\, 1), \end{cases} \tag{1.2}$$

then

$$(-\Delta)^\alpha |x|^{\bar{\tau}} + \mu_0 |x|^{\bar{\tau} - 2\alpha} = 0, \qquad \forall\, x \in \mathbb{R}^N \setminus \{0\}.$$

For $\mu > \mu_0$, there exists a unique $\tau_\alpha(\mu) \in (-N,\, \bar{\tau})$ such that

$$\phi_{\tau_\alpha(\mu)}(x) := |x|^{\tau_\alpha(\mu)} \tag{1.3}$$

is a fundamental solution of $(-\Delta)^\alpha + \frac{\mu}{|x|^{2\alpha}}$, i.e.

$$(-\Delta)^\alpha \phi_{\tau_\alpha(\mu)} + \frac{\mu}{|x|^{2\alpha}} \phi_{\tau_\alpha(\mu)} = 0 \quad \textit{in} \quad \mathbb{R}^N \setminus \{0\}. \tag{1.4}$$

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