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Anisotropic variable Hardy–Lorentz spaces and their real interpolation ☆

Jun Liu, Dachun Yang *, Wen Yuan

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

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A B S T R A C T

Let $p(\cdot):\ \mathbb{R}^n \to (0,\infty]$ be a variable exponent function satisfying the globally log-Hölder continuous condition, $q \in (0,\infty]$ and A be a general expansive matrix on $\mathbb{R}^n$. In this article, the authors first introduce the anisotropic variable Hardy–Lorentz space $H_A^{p(\cdot),q}(\mathbb{R}^n)$ associated with A, via the radial grand maximal function, and then establish its radial or non-tangential maximal function characterizations. Moreover, the authors also obtain characterizations of $H_A^{p(\cdot),q}(\mathbb{R}^n)$, respectively, in terms of the atom and the Lusin area function. As an application, the authors prove that the anisotropic variable Hardy–Lorentz space $H_A^{p(\cdot),q}(\mathbb{R}^n)$ serves as the intermediate space between the anisotropic variable Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ via the real interpolation. This, together with a special case of the real interpolation theorem of H. Kempka and J. Vybíral on the variable Lorentz space, further implies the coincidence between $H_A^{p(\cdot),q}(\mathbb{R}^n)$ and the variable Lorentz space $L^{p(\cdot),q}(\mathbb{R}^n)$ when $\operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \in (1,\infty)$.

1. Introduction

As a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$, the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, in which the constant exponent p is replaced by an exponent function $p(\cdot):\ \mathbb{R}^n \to (0,\infty]$, were studied by Musielak [54] and Nakano [56,57], which can be traced back to Orlicz [60,61]. But the modern theory of function spaces with variable exponents was started with the articles [45] of Kováčik and Rákosník and [32] of Fan and Zhao as well as [21] of Cruz-Uribe and [24] of Diening, and nowadays has been widely used in harmonic analysis (see, for example, [22,25,80]). In addition, the theory of variable function spaces also has interesting applications in fluid dynamics [2], image processing [17], partial differential equations and variational calculus [3,30,40,66].

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* Corresponding author.
E-mail addresses: junliu@mail.bnu.edu.cn (J. Liu), dcyang@bnu.edu.cn (D. Yang), wenyuan@bnu.edu.cn (W. Yuan).

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Recently, Nakai and Sawano [55] and, independently, Cruz-Uribe and Wang [23] with some weaker assumptions on $p(\cdot)$ than those used in [55], extended the theory of variable Lebesgue spaces via investigating the variable Hardy spaces on $\mathbb{R}^n$. Later, Sawano [69], Zhuo et al. [86] and Yang et al. [84] further completed the theory of these variable Hardy spaces. For more developments of function spaces with variable exponents, we refer the reader to [6,26,44,58,59,76–79,83] and their references. In particular, Kempka and Vybíral [44] introduced the variable Lorentz spaces which are a generalization of both the variable Lebesgue spaces and the classical Lorentz spaces and obtained some basic properties of these spaces including several embedding conclusions. The real interpolation result that the variable Lorentz space serves as the intermediate space between the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ was also presented in [44].

Very recently, Yan et al. [79] first introduced the variable weak Hardy spaces on $\mathbb{R}^n$ and established their various real-variable characterizations; as an application, the boundedness of some Calderón–Zygmund operators on these spaces in the critical case was also presented. Based on these results, via establishing a very interesting decomposition for any distribution of the variable weak Hardy space, Zhuo et al. [87] proved the following real interpolation theorem between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$:

$$(H^{p(\cdot)}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,\infty} = WH^{p(\cdot)/(1-\theta)}(\mathbb{R}^n), \quad \theta \in (0,1), \tag{1.1}$$

where $WH^{p(\cdot)/(1-\theta)}(\mathbb{R}^n)$ denotes the variable weak Hardy space and $(\cdot,\cdot)_{\theta,\infty}$ the real interpolation.

As was well known, Fefferman et al. [34] showed that the Hardy–Lorentz space $H^{p,q}(\mathbb{R}^n)$ was actually the intermediate space between the classical Hardy space $H^p(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ under the real interpolation, which is the main motivation to develop the real-variable theory of $H^{p,q}(\mathbb{R}^n)$. Thus, it is natural and interesting to ask whether or not the variable Hardy–Lorentz space also serves as the intermediate space between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ via the real interpolation, namely, if $(\cdot,\cdot)_{\theta,\infty}$ in (1.1) is replaced by $(\cdot,\cdot)_{\theta,q}$ with $q \in (0,\infty]$, what happens?

On the other hand, as the series of works (see, for example, [1,5,7,34,35,49,62]) reveal, the Hardy–Lorentz spaces (as well the weak Hardy spaces) serve as a more subtle research object than the usual Hardy spaces when studying the boundedness of singular integrals, especially, in some critical cases, due to the fact that these function spaces own finer structures. Moreover, after the celebrated articles [14–16] of Calderón and Torchinsky on parabolic Hardy spaces, there has been an enormous interest in extending classical function spaces arising in harmonic analysis from Euclidean spaces to some more general underlying spaces; see, for example, [28,36,38,39,70,67,68,72,73,81]. The function spaces in the anisotropic setting have proved of wide generality (see, for example, [10–12]), which include the classical isotropic spaces and the parabolic spaces as special cases. For more progresses about this theory, we refer the reader to [46,47,51–53,31,74,75] and their references. In particular, the authors recently introduced the anisotropic Hardy–Lorentz spaces $H^{p,q}_A(\mathbb{R}^n)$, associated with some dilation A, and obtained their various real-variable characterizations (see [51,52]). Also, very recently, Zhuo et al. [85] developed the real-variable theory of the variable Hardy space $H^{p(\cdot)}(\mathcal{X})$ on an RD-space $\mathcal{X}$. Recall that a metric measure space of homogeneous type $\mathcal{X}$ is called an RD-space if it is a metric measure space of homogeneous type in the sense of Coifman and Weiss [19,20] and satisfies some reverse doubling property, which was originally introduced by Han et al. [39] (see also [82] for some equivalent characterizations).

To further study the intermediate space between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ via the real interpolation and also to give a complete theory of variable Hardy–Lorentz spaces in anisotropic setting, in this article, we first introduce the anisotropic variable Hardy–Lorentz space, via the radial grand maximal function, and then establish its several real-variable characterizations, respectively, in terms of the atom, the radial or the non-tangential maximal functions, and the Lusin area function. As an application, we prove that the anisotropic variable Hardy–Lorentz space $H^{p(\cdot),q}_A(\mathbb{R}^n)$ serves as the intermediate space between the anisotropic variable Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ via the