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Operator-valued operators that are associated to vector-valued operators

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This paper is motivated by a long-standing conjecture of Dinculeanu from 1967. Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. Dinculeanu conjectured that there exist operators $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ which are not associated to any $U \in \mathcal{L}(\mathcal{C}(\Omega,X),Y)$. We study this existence problem systematically on three possible levels of generality: the classical case $\mathcal{C}(\Omega,X)$ of continuous vector-valued functions, p-continuous vector-valued functions, and tensor products. On each level, we establish necessary and sufficient conditions for an $\mathcal{L}(X,Y)$-valued operator to be associated to a Y-valued operator. Among others, we see that examples, proving Dinculeanu's conjecture, come out on the all three levels of generality.

1. Introduction

Let X and Y be Banach spaces and let Ω be a compact Hausdorff space. The space of continuous functions from Ω into X ($\mathbb{K}$, respectively) is denoted by $\mathcal{C}(\Omega,X)$ ($\mathcal{C}(\Omega)$, respectively). Let $\mathcal{L}(X,Y)$ denote the Banach space of bounded linear operators from X into Y. For every operator $U \in \mathcal{L}(\mathcal{C}(\Omega,X),Y)$, we denote by $U^\#$ the *associated operator* from $\mathcal{C}(\Omega)$ to $\mathcal{L}(X,Y)$ defined by $(U^\#\varphi)x = U(\varphi x)$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. (The notation $U^\#$ is traditional; see, e.g., [15,21–24,26,29].) Then, clearly, $U^\# \in \mathcal{L}(\mathcal{C}(\Omega),\mathcal{L}(X,Y))$.

On the other hand, in a short remark (see [6, Remark, p. 379]), Dinculeanu pointed out that there exist operators $S \in \mathcal{L}(\mathcal{C}(\Omega),\mathcal{L}(X,Y))$ which are not associated to any $U \in \mathcal{L}(\mathcal{C}(\Omega,X),Y)$, meaning that $S \neq U^\#$ for all operators $U \in \mathcal{L}(\mathcal{C}(\Omega,X),Y)$. (In [6], $U^\#$ is denoted by U'.) Professor Dinculeanu kindly informed us (personal communication, September 27, 2015) that his remark was just a conjecture based on Grothendieck's result quoted in Remark 2.4 below.

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Many authors have studied the interplay between U and $U^{\#}$ for different classes of operators (see, e.g., the above references). However, it seems that nothing (apart from Dinculeanu's remark) has been said about the problem of the existence of an operator U such that $U^{\#} = S$ for a given operator S.

This paper aims in studying this existence problem and, in particular, in proving Dinculeanu's conjecture. However, we shall study the problem in a more general context of operators defined on the Banach space $\mathcal{C}_p(\Omega, X)$ of p-continuous X-valued functions (see Section 2 for the definition and references). Since $\mathcal{C}_\infty(\Omega, X) = \mathcal{C}(\Omega, X)$, this also encompasses the classical case of operators on $\mathcal{C}(\Omega, X)$.

By Grothendieck's classics [11] (see, e.g., [25, pp. 49–50]), we know that

$$\mathcal{C}(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_\varepsilon X,$$

where ε denotes the injective tensor norm, under the canonical isometric isomorphism $\varphi x \leftrightarrow \varphi \otimes x$, $\varphi \in \mathcal{C}(\Omega)$ and $x \in X$. As is well known, this allows to extend the definition of $U^{\#}$ as follows.

Let Z be a Banach space and let α be a tensor norm. If $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$, then the operator $U^{\#} \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ *associated to* U is defined by $(U^{\#}z)x = U(z \otimes x)$, $z \in Z$ and $x \in X$. By a recent result of the authors [16, Theorem 3.8],

$$\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X,$$

where d_p denotes the right Chevet–Saphar tensor norm (see [27] or, e.g., [25, Chapter 6]). Keeping this in mind, we shall study the existence problem in the general context of operators defined on tensor products of Banach spaces. In particular, we shall see that examples, proving Dinculeanu's conjecture, come out on the all three levels of the generality (see Remarks 2.2 and 2.4, Corollary 3.4, Proposition 4.4).

Let Z, X, and α be as above. In Section 2, we prove a general omnibus theorem (Theorem 2.1), which provides three equivalent conditions for the existence of $U \in \mathcal{L}(Z\hat{\otimes}_\alpha X, Y)$ such that $S = U^{\#}$ for every Banach space Y and every operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$. The main applications (Theorem 2.3 and Corollaries 2.6 and 2.7) concern the case of p-continuous X-valued functions $\mathcal{C}_p(\Omega, X) = \mathcal{C}(\Omega)\hat{\otimes}_{d_p} X$.

In Section 3, we fix Banach spaces Z, X, and Y, and a tensor norm d_p. We prove another omnibus theorem (Theorem 3.3), which provides four equivalent conditions for a given operator $S \in \mathcal{L}(Z, \mathcal{L}(X, Y))$ to be the associated operator to an operator $U \in \mathcal{L}(Z\hat{\otimes}_{d_p} X, Y)$. Again, the main application (Corollary 3.4) concerns $\mathcal{C}_p(\Omega, X)$, yielding conditions that seem to be new even in the classical case $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$.

In Section 4, we are given an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$. We present a necessary condition for the existence of $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ such that $S = U^{\#}$ (Proposition 4.2), which becomes also sufficient in the case $\mathcal{C}(\Omega, X) = \mathcal{C}_\infty(\Omega, X)$ (Proposition 4.4). This condition is expressed in terms of the representing measure of S, so we build the representing measure of such kind of operators.

Section 5 provides three examples (concerning Corollaries 2.7, 3.5, and Proposition 4.2) in order to show that our results are sharp in general.

Our notation is standard. Let $1 \le p \le \infty$, and denote by p' the conjugate index of p (i.e., $1/p + 1/p' = 1$ with the convention $1/\infty = 0$). We consider Banach spaces over the same, either real or complex, field $\mathbb{K}$. The closed unit ball of X is denoted by B_X. The Banach space of all *absolutely p-summable sequences* in X is denoted by $\ell_p(X)$ and its norm by $\|\cdot\|_p$. By $\ell_p^w(X)$ we mean the Banach space of *weakly p-summable sequences* in X with the norm $\|\cdot\|_p^w$ (see, e.g., [4, pp. 32–33]). Denote by $\ell_p^u(X)$ the Banach space of all *unconditionally p-summable sequences* in X, which is the closed subspace of $\ell_p^w(X)$ formed by the sequences $(x_n) \in \ell_p^w(X)$ satisfying $(x_n) = \lim_{N\to\infty}(x_1, \ldots, x_N, 0, 0, \ldots)$ in $\ell_p^w(X)$ (see [9] or, e.g., [3, 8.2, 8.3]). We refer to Pietsch's book [19] for the theory of operator ideals and, in particular, to the book [4] by Diestel, Jarchow, and Tonge for absolutely (r, q)- and q-summing operators. Our main reference on the theory of tensor norms and related Banach operator ideals is the book of Ryan [25].

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