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Invariant measures for continued fraction algorithms with finitely many digits

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ABSTRACT

In this paper we consider continued fraction (CF) expansions on intervals different from $[0, 1]$. For every x in such interval we find a CF expansion with a finite number of possible digits. Using the natural extension, the density of the invariant measure is obtained in a number of examples. In case this method does not work, a Gauss–Kuzmin–Lévy based approximation method is used. Convergence of this method follows from [32] but the speed of convergence remains unknown. For a lot of known densities the method gives a very good approximation in a low number of iterations. Finally, a subfamily of the N-expansions is studied. In particular, the entropy as a function of a parameter α is estimated for $N = 2$ and $N = 36$. Interesting behavior can be observed from numerical results.

1. Introduction

In general, studies on continued fraction expansions focus on expansions for which almost all x[1] have an expansion with digits from an infinite alphabet. A classical example is the regular continued fraction [14,17,31]. An example of continued fraction expansions with only finitely many digits has been introduced in [28] by Joe Lehner, where the only possible digits are 1 and 2; see also [13]. More recently, continued fractions have been investigated for which all x in a certain interval have finitely many possible digits. In [16] the following 4-expansion has been (briefly) studied. Let $T : [1, 2] \to [1, 2]$ be defined as (see Fig. 1)

$$T(x) = \begin{cases} \dfrac{4}{x} - 1 & \text{for} \quad x \in (\frac{4}{3}, 2] \\ \dfrac{4}{x} - 2 & \text{for} \quad x \in [1, \frac{4}{3}] \ . \end{cases} \tag{1}$$

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[1] All ‘almost all x’ statements are wrt. Lebesgue measure.

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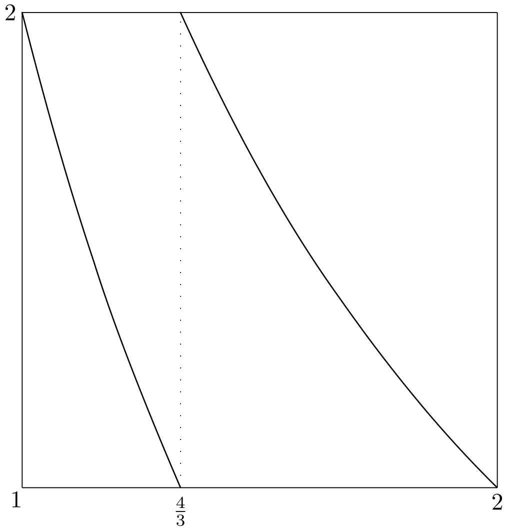

Fig. 1. The CF-map T from (1).

By repeatedly using this map we find that every $x \in [1, 2]$ has an infinite continued fraction expansion of the form

$$x = \cfrac{4}{d_1 + \cfrac{4}{d_2 + \ddots}} \tag{2}$$

with $d_n \in \{1, 2\}$ for all $n \geq 1$. The class of continued fractions algorithms that give rise to digits from a finite alphabet is very large. In this introductory paper we will give examples of such expansions. Most of the examples will be a particular case of N-expansions (see [1,3,16]). Other examples are closely related and can be found by combining the N-expansions with flipped expansions (cf. [26] for 2-expansions; see also [11] for flipped expansions). For all these examples we refer to [11] for ergodicity (which can be obtained in all these cases in a similar way) and existence of an invariant measure. In a number of cases however, it is difficult to find the invariant measure explicitly, while in seemingly closely related cases it is very easy. In case we cannot give an analytic expression for the invariant measure we will give an approximation using a method that is very suitable (from a computational point of view) for expansions with finitely many different digits. This method is based on a Gauss–Kuzmin–Lévy Theorem. For our method, convergence follows from [32]. For greedy N-expansions this theorem is proved by Dan Lascu in [27]. The method yields smoother results than by simulating in the classical way (looking at the histogram of the orbit of a typical point as described in Geon Choe's book [9], and used in his papers [8,10]). We also give an example in which we do know the density and where we use this method to show its strength.

In Section 2 we will give the general form of the continued fraction maps we study in this paper. After that we give several examples of such maps and a way of finding the density of the invariant measure by using the natural extension. In Section 2.2 we will see how we simulated the densities in case we were not able to find them explicitly. In the last section we will consider a subfamily of the N-expansions which can be parameterized by $\alpha \in (0, \sqrt{N} - 1]$. We study the entropy as function of α. We will do so mainly on a numerical basis. In the past decades, it turned out to be very interesting to look at entropy of a family of continued fractions as a function of a parameter. In [2,5–7,25,29,30] the entropy of Nakada's α-expansions is studied. For example in [30] it is shown that in any neighborhood of 0 you can always find an interval on which the entropy function is increasing, an interval on which the entropy function is decreasing and an interval on which this function is constant. In [6] it is shown that there is a countable set of open intervals on