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On the scaling methods by Pinchuk and Frankel ☆

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ABSTRACT

The main purpose of this paper is to study two scaling methods developed respectively by Pinchuk and Frankel. We introduce first a continuously-varying global coordinate system, and give an alternative proof to the convergence of Pinchuk's scaling sequence (and of our modification) on bounded domains with finite type boundaries in $\mathbb{C}^2$. Using this, we discuss the modification of the Frankel scaling sequence on nonconvex domains. We also observe that two modified scalings are equivalent.

1. Introduction

The scaling methods were introduced by Pinchuk [11] and Frankel [7] independently in the 1980's as a technique to study bounded domains in $\mathbb{C}^n$ with noncompact automorphism group. These techniques have been developed further by many authors and have become an important tool to prove the results of [11,1,9] and others.

Pinchuk's scaling sequence was constructed by a sequence of compositions of stretching maps, say Λ_j, and automorphisms ϕ_j of the given domain Ω. If the automorphism group of Ω is noncompact then, in many cases, there is a sequence $\{\phi_j\} \subset Aut(\Omega)$ which contracts compact subsets successively to some boundary point. On the other hand, a sequence of stretching maps is a divergent sequence of shear maps, the composition of $\mathbb{C}$-affine maps and triangular maps. The general expectation is that there is a subsequence of the Pinchuk scaling sequence $\{\sigma_j := \Lambda_j \circ \phi_j\}$ convergent to the limit map, say $\widehat{\sigma}$, uniformly on compact subsets of Ω. If this limit were 1-1, then it would be a re-embedding of Ω into $\mathbb{C}^n$. If Ω is a domain in $\mathbb{C}$ with smooth boundary, then the image of the limit map turns out to be a half plane. This is the special case of the Riemann mapping theorem and therefore it seems natural to hope for the convergence of $\{\sigma_j\}$ in all dimensions.

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As the first result in the higher dimensions, Pinchuk proved that his scaling sequence has a subsequence that converges to a biholomorphism uniformly on compact subsets for the class of bounded strongly pseudoconvex domains in all dimensions [11]; this proves the Wong–Rosay theorem [14,12]. And later, Bedford and Pinchuk [1] showed the convergence of the sequence if the domain is bounded with a finite type boundary in the sense of D'Angelo [5].

One of the difficulties in proving the convergence is that the expected limit domain $\widehat{\Omega}$ is not bounded; its Kobayashi hyperbolicity is not *a priori* clear. Pinchuk considers, alternatively, the convergence of the backward scaling sequence $\{\sigma_j^{-1}\}$. If the limit domain $\widehat{\Omega}$ is well-defined in some sense, then this sequence $\{\sigma_j^{-1}\}$ always has a convergent subsequence by Montel's theorem. Now, a question arises naturally: is the limit map of the backward sequence 1-1? For the class of bounded domains in $\mathbb{C}^2$ whose boundaries are of finite type, Bedford and Pinchuk have given a general affirmative resolution [1] (cf. [3] also). If the limit map is surjective, additionally, then it follows that the inverse of the limit map is actually the same as a subsequential limit of the initial Pinchuk scaling sequence. This establishes the Pinchuk scaling method.

The Frankel scaling sequence follows the same principle but its construction is different. Given a domain Ω, a point $p \in \Omega$ and a sequence of automorphisms $\{\phi_j\}$, it is defined directly by $\omega_j(z) := [d\phi_j|_p]^{-1}(\phi_j(z) - \phi_j(p))$. If $\{\phi_j(p)\}$ converges to some boundary point of Ω, then $\{\phi_j\}$ cannot converge to another automorphism. In fact, $\lim_{j\to\infty} \det(d\phi_j|_p) = 0$. Then $[d\phi_j|_p]^{-1}$ diverges. So, Frankel's scaling method appears to be similar to Pinchuk's. The sequence $\{[d\phi_j|_p]^{-1}\}$ stretches in some sense, whereas the sequence $\{\phi_j\}$ contracts. Now one can naturally pose the question: when does Frankel's scaling sequence form a normal family? Frankel proved that it suffices for Ω to be convex and Kobayashi hyperbolic [7].

The purpose of this article is summarised as follows:

In Section 2, we introduce a special continuously-varying coordinate system, pertaining to the target boundary point. Using this coordinate system, we give another proof of the convergence of the Pinchuk scaling sequence on a bounded domain in $\mathbb{C}^2$ with smooth finite-type boundary. We feel that our proof is simpler and more straightforward than that of Berteloot–Cœuré [3].

Section 3 concerns the Frankel scaling sequence. The convexity was essential for its convergence to a holomorphic embedding into $\mathbb{C}^n$. There has been a question whether it converges without convexity. Here, we give a modification of the Frankel scaling sequence so that they may converge also on some nonconvex domains, using a sequence $\{\psi_j\}$ of automorphisms of $\mathbb{C}^n$ that converges to another. Two examples are given to show several aspects of the (modified) Frankel scaling sequence.

Finally in Section 4, we observe that the limit maps, if they exist, of Pinchuk and modified Frankel's scaling sequences are equivalent. Notice that this generalizes a theorem of Kim/Krantz in [10] for the convex case.

In the Appendix, we prove the existence of a coordinate system introduced in Section 2.

2. The Pinchuk scaling sequence

Recall the definition of finite type in the sense of D'Angelo [5].

Definition 2.1. Let Ω be a domain in $\mathbb{C}^n$ with smooth boundary. Let q be a point in $\partial\Omega$ and ρ be a local defining function of Ω at q. The *type* $\Delta(q) = \Delta(\Omega, q)$ at q is the positive value defined by

$$\Delta(q) := \sup_h \frac{\nu(\rho \circ h)}{\nu(h - q)},$$

where $\nu(f)$ is the order of vanishing of f at 0 and the supremum is taken over all nontrivial analytic discs h in $\mathbb{C}^n$ with $h(0) = q$.

The point q is called a *finite type boundary point* of Ω if $\Delta(q)$ is finite. If all the boundary points of Ω is of finite type, then Ω is called a *domain with finite type boundary.*