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Weighted functional spaces approach in infinite horizon optimal control problems: A systematic analysis of hidden opportunities and advantages ☆

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A R T I C L E I N F O

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A B S T R A C T

This paper constitutes the advantages of using the weighted Sobolev and weighted Lebesgue spaces in optimal control problems defined on an infinite time interval. Based on numerous examples, it demonstrates how the introduction of certain weight or density functions into the problem statement may influence the existence of an optimal solution and the optimality of a chosen candidate. The positive effects of the proper relation between the state and the co-state functional spaces for the necessary optimality conditions as well as for the development of numerical schemes are discussed. This paper is based on over one decade of research in this field and summarizes the main findings concerning the use of weight functions in optimal control problems.

1. Introduction

Optimal control problems with infinite horizon arise from different points of view. On the one hand, considering a control problem with a long, but not bounded, finite horizon T leads to the limiting process $T \to \infty$ and to the infinite horizon $T = \infty$ as the idealization of this limiting procedure. In this sense, most economic applications are studied and most of the theoretical results follow from these ideas.

On the other hand, let us mention the second sort of ideas where the infinite horizon a priori appears. The concept of designing a feedback controller such that an integral of the square of the tracking error over an infinite horizon is minimized was first proposed by Wiener [42], 1943, and Hall [17], 1949. The conventional theory of the regulator problem in [17,42] is based on Fourier and Laplace transforms and is restricted to the case of autonomous problems. In [21], 1960, Kalman remarked that the problem class is quite broad and

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there are many unsettled questions. He proposed another approach to the problem: In his ground-breaking paper he recommended to use the well known theory of ordinary differential equations and optimal control of a receding horizon regulator problem, see e.g. [20], to design a feedback control law.

However, the integration over an unbounded time interval indicates some difficulties which are specific to this class of problems. Such problematic features contain, for instance, the lack of a standard transversality condition which results in a missing boundary condition in the system of necessary optimality conditions. This, in turn, leads to extreme sensitivity of numerical procedures making it even impossible to find a solution. Another substantial problem is the non-compactness of the time domain, which makes serious problems in the proofs of existence results due to non-extendible embedding properties of the underlying spaces onto the unbounded domain. For this reason, one needs other techniques to cope with typical difficulties arising in infinite horizon control problems. The suggestion of introducing weighted functional spaces, made in [37] holds many interesting effects and advantages both for the modelling itself and the theoretical and numerical treatment of the control problem. The systematic analysis and discussion of these effects demonstrated by means of enlightening examples represents the main focus of this paper.

We also would like to mention that in the paper of Aubin and Clarke, cf. [5], weighted measures and duality concept were already used on the space of locally absolutely continuous functions over the half line for control problems with linear dynamics and constant coefficients. But the proof of Theorem 1, and Lemma 3.1 in their paper use implicitly the property that the state variable belongs, due to made assumptions, to a weighted Sobolev space without mentioning this notion explicitly. The usage of this property results in corresponding transversality conditions for the adjoint and its derivative. This transversality condition is given in form of finiteness of certain integrals over the half line. Restricting our results concerning Pontryagin Type Maximum Principle onto the class of autonomous control problems with linear dynamics would lead to results comparable to those of [5], i.e. our results, e.g. Theorem 2, represent generalization of Theorem 1 in [5].

Our paper is organized as follows. In the second section, the main specific problematic points of infinite horizon optimal control problems are described. Section 3 contains the definitions of the weighted functional spaces and the main useful properties from the functional analytical point of view. Section 4 is devoted to discussion of the degrees of freedom in the modelling provided by the weighted functional spaces approach as well as contains the general problem statement. Section 5 describes the "proper" relation between the functional space for the state trajectories and the space for the co-state trajectories. Here we also give examples showing that this relation is very often violated in the literature as well as examples showing that it is not sufficient and not satisfactory to choose the non-weighted functional spaces instead of weighted. Section 7 investigates the interplay between the state and co-state functional spaces and the influence of its relation on the validity of the necessary transversality condition in the Pontryagin type Maximum principle. Which advantages we may obtain for the development of fast procedures for numerical solution by applying the weighted functional spaces approach is discussed in section 8. In Section 9, we deal with considering the weighted local optimality notion and opportunities opened by this approach.

2. Typical difficulties in infinite horizon problems

2.1. Dependence of the solution on the used integral notion

We consider the objective functional as an integral of a function g over an infinite horizon $[0, \infty)$. All integrations hereafter will mean Lebesgue integration. A function g is said to be summable, if the integral of the non-negative part $g_+ := \max\{g, 0\}$ and the non-positive part $g_- := \max\{-g, 0\}$ are finite; the function g is said to be integrable, if at least one of these integrals is finite. The **proper** Lebesgue integral of g is

$$I_\infty := \int_0^\infty g(t)dt := \int_0^\infty g_+(t)dt - \int_0^\infty g_-(t)dt$$