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Structure and stability of the equilibrium set in potential-driven flow networks ☆

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ABSTRACT

In this paper we address local bifurcation properties of a family of networked dynamical systems, specifically those defined by a potential-driven flow on a (directed) graph. These network flows include linear consensus dynamics or Kuramoto models of coupled nonlinear oscillators as particular cases. As it is well-known for consensus systems, these networks exhibit a somehow unconventional dynamical feature, namely, the existence of a line of equilibria, following from a well-known property of the graph Laplacian matrix in connected networks with positive weights. Negative weights, which arise in different contexts (e.g. in consensus models in signed graphs or in Kuramoto models with antagonistic actors), may on the one hand lead to higher-dimensional manifolds of equilibria and, on the other, be responsible for bifurcation phenomena. In this direction, we prove a saddle-node bifurcation theorem for a broad family of potential-driven flows, in networks with one or more negative weights. The goal is to state the conditions in structural terms, that is, in terms of the expressions defining the flowrates and the graph-theoretic properties of the network. Not only the eigenvalue requirements but also the nonlinear transversality assumptions supporting the bifurcation motivate an analysis of independent interest concerning the rank degeneracies of nodal matrices arising in the linearized dynamics; this analysis is performed in terms of the contraction–deletion structure of spanning trees and uses several results from matrix analysis. Different examples illustrate the results; some linear problems (including signed graphs) are aimed at illustrating the analysis of nodal matrices, whereas in a nonlinear framework we apply the characterization of saddle-node bifurcations to networks with a sinusoidal (Kuramoto-like) flow.

1. Introduction

This paper addresses certain qualitative properties of potential-driven flows defined on networks. From different perspectives, this kind of problems has been addressed in many application fields, which include

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coupled oscillators, operations research, circuit theory, electronic engineering, water and gas networks, power systems, traffic networks or multiagent systems, to name but a few: cf. [2,3,9–11,16,21,26–28,32,34,37,42,48]. Our point of view is deliberately a general one, as we aim to explore in structural terms certain properties of dynamical systems defined on networks, without focusing on specific models coming from applications; in our analysis we examine systematically the way in which the dynamical processes interact with the graph-theoretic underlying structure, looking for general results except for the fact that the dynamics is assumed to be defined by a potential-driven flow. Very broadly speaking, this approach has been systematically exploited in nonlinear circuit theory in the last decade [29,39]: our present goal is to extend somehow this perspective to the study of certain aspects of complex network dynamics, which define a very active topic research (see the works cited above and references therein). From this point of view the scope of potential applications of our results is rather wide.

Specifically, we are interested in the fact that potential-driven flows systematically yield non-isolated equilibrium points. This is well-known for instance in the context of so-called consensus protocols [34,37,44], where *lines* of equilibria arise pervasively. These consensus protocols can be easily reformulated as a potential-driven flow in a network of cooperating agents: find details in Section 2. The one-dimensional nature of the equilibrium set can be understood as a consequence of the structure of the (positively) weighted Laplacian matrices arising in these models; here the positive weights reflect the fact that agents cooperate in the sense that they tend to reach a common position; alternatively, this model can be seen as a redistribution system in which a flow evolves in a way such that all agents asymptotically get the same quantity of a given commodity or resource.

Our goal is to examine what happens, from the point of view of local dynamics, when this positiveness (or cooperation) assumption fails and weights can take on negative values. Negatively weighted networks have received some recent attention as a model of systems with *antagonistic* actors (cf. [3,23,33,45]). Specifically, the presence of negative weights should be responsible for stability changes and, in a nonlinear setting, bifurcation theory is the appropriate context to frame local qualitative changes near equilibria. In this direction, our main result will be a saddle-node bifurcation theorem for potential-driven flows in networks, holding in contexts with either one or several negative weights. The non-isolated nature of equilibria will yield a splitting of lines of equilibria displaying different stability properties. These results will be presented in Section 4. However, the eigenvalue requirements as well as the transversality conditions supporting the saddle-node bifurcation theorem for general ODEs [20,36,41] pose certain problems of independent interest regarding the (negatively weighted) nodal matrices which arise in the linearized dynamics; for this reason, we tackle in advance the structure of the equilibrium set and several related properties of independent interest involving nodal matrices in Section 3, using determinantal expansions as in the weighted matrix-tree theorem; these results, whose origins go back to the pioneering work of Kirchhoff and Maxwell, will be placed in the appropriate context by examining additionally several related recent works [6,9,10,13,46,47].

Potential-driven flow networks include in particular a wide family of Kuramoto models (cf. [2,16,23,26,28,42,45]) and we will show, specifically, that our saddle-node bifurcation characterization applies to certain networks with a sinusoidal (Kuramoto-like) flow, extending some results from [16]. This, together with some additional examples concerning flows in signed graphs, can be found in Section 5. Finally, Section 6 compiles concluding remarks and lines for future research.

2. Potential-driven flows in networks

Networks will be assumed in this paper to be defined by a directed graph (or digraph) without self-loops. For the sake of simplicity we assume throughout the document that the digraph is connected and that it has at least one edge. Denoting by n and m the number of nodes and edges, respectively, define the incidence matrix $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ entrywise by