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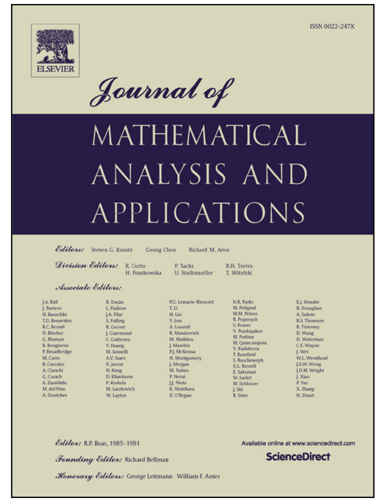

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P.A. Martin

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Bounds on ratios of modified Bessel functions with complex arguments

P. A. Martin

Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401-1887, USA

Abstract

A simple uniform bound for the ratio of two modified spherical Bessel functions is derived. The arguments of the two functions are complex but their ratio is real.

Keywords: spherical Bessel function, Bessel polynomial, generalized hypergeometric function, special function

1. Introduction

In a recent investigation in the context of acoustic scattering, we encountered a Volterra integral equation of the second kind,

$$u(x) - \int_x^\infty \mathcal{K}(x,y)\, u(y)\, \mathrm{d}y = f(x), \quad x > 1, \tag{1}$$

where $f(x)$ is given and $u(x)$ is to be found. The kernel is given by

$$\mathcal{K}(x,y) = \int_x^y \left(\frac{W(y)}{W(\eta)}\right)^2 \mathrm{d}\eta. \tag{2}$$

where

$$W(\xi) = \xi^{1/2} K_\nu(\mu\xi), \tag{3}$$

K_ν is a modified Bessel function, and the parameters μ and ν will be specified shortly. Usually, Volterra integral equations of the second kind (such as (1)) can be solved by iteration. To justify this approach, a bound on $|\mathcal{K}(x,y)|$ is needed.

Fortunately, there is a recent paper by Baricz [1] containing a thorough review of the literature on known bounds on ratios of modified Bessel functions. From [1, eqn (3.6)],

$$\frac{K_\nu(x)}{K_\nu(y)} > \mathrm{e}^{y-x}\left(\frac{y}{x}\right)^{1/2}, \quad |\nu| > \frac{1}{2}, \quad 0 < x < y.$$

Hence

$$F(y,\eta) \equiv \frac{W(y)}{W(\eta)} = \frac{y^{1/2}K_\nu(\mu y)}{\eta^{1/2}K_\nu(\mu\eta)} < \mathrm{e}^{\mu(\eta-y)}, \quad 0 < \eta < y. \tag{4}$$

This holds for real μ with $\mu > 0$. Using the bound (4) in (2),

$$0 < \mathcal{K}(x,y) < \int_x^y \mathrm{e}^{2\mu(\eta-y)}\, \mathrm{d}\eta = \frac{1}{2\mu}\left(1 - \mathrm{e}^{-2\mu(y-x)}\right) \le \frac{1}{2\mu},$$

as $y \ge x$, and this uniform bound can be used to justify an iterative scheme.

Email address: pamartin@mines.edu (P. A. Martin)