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Asymptotic behaviour of solutions of the fast diffusion equation near its extinction time

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ABSTRACT

Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\beta \geq \frac{m\rho_1}{n-2-nm}$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. For any $\lambda > 0$, we will prove the existence and uniqueness (for $\beta \geq \frac{\rho_1}{n-2-nm}$) of radially symmetric singular solution $g_\lambda \in C^\infty(\mathbb{R}^n \setminus \{0\})$ of the elliptic equation $\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0$, $v > 0$, in $\mathbb{R}^n \setminus \{0\}$, satisfying $\lim_{|x|\to 0} |x|^{\alpha/\beta} g_\lambda(x) = \lambda^{-\frac{\rho_1}{(1-m)\beta}}$. When β is sufficiently large, we prove the higher order asymptotic behaviour of radially symmetric solutions of the above elliptic equation as $|x| \to \infty$. We also obtain an inversion formula for the radially symmetric solution of the above equation. As a consequence we will prove the extinction behaviour of the solution u of the fast diffusion equation $u_t = \Delta u^m$ in $\mathbb{R}^n \times (0, T)$ near the extinction time $T > 0$.

1. Introduction

The equation

$$u_t = \Delta u^m \tag{1.1}$$

appears in many physical models. When $m > 1$, (1.1) is the porous medium equation which models the flow of gases or liquid through porous media. When $m = 1$, (1.1) is the heat equation. When $0 < m < 1$, (1.1) is the fast diffusion equation. When $m = \frac{n-2}{n+2}$, $n \geq 3$, and $g = u^{\frac{4}{n+2}} dx^2$ is a metric on $\mathbb{R}^n$ which evolves by the Yamabe flow,

$$\frac{\partial g}{\partial t} = -Rg$$

where R is the scalar curvature of the metric g, then u satisfies [5,19],

$$u_t = \frac{n-1}{m} \Delta u^m.$$

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Because of the importance of the equation (1.1) and its relation to the Yamabe flow, a lot of researches on this topic appear recently, see for example, works of P. Daskalopoulos, J. King, M. del Pino, N. Sesum, M. Sáez [5–8,19], S.Y. Hsu [13–15], K.M. Hui [16,17], M. Fila, J.L. Vazquez, M. Winkler, E. Yanagida [9,10], A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J.L. Vazquez [2,3], etc. We refer the reader to the survey paper [1] by D.G. Aronson and the books [4,21], by P. Daskalopoulos, C.E. Kenig, and J.L. Vazquez on the recent progress on this equation.

As observed by J.L. Vazquez [20], M.A. Herrero and M. Pierre [12], and others [14,16], there is a big difference in the behaviour of solution of (1.1) for the case $\frac{n-2}{n} < m < 1$, $n \ge 3$, and the case $0 < m \le \frac{n-2}{n}$, $n \ge 3$. For example for any $0 \le u_0 \in L^1_{loc}(\mathbb{R}^n)$, $u_0 \not\equiv 0$, when $\frac{n-2}{n} < m < 1$, $n \ge 3$, there exists a unique global positive smooth solution of (1.1) in $\mathbb{R}^n \times (0,\infty)$ with initial value u_0 [12]. On the other hand when $0 < m < \frac{n-2}{n}$, $n \ge 3$, there exists $0 \le u_0 \in L^1_{loc}(\mathbb{R}^n)$, $u_0 \not\equiv 0$, and $T > 0$ such that the solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \tag{1.2}$$

extincts at time T [7]. Since the asymptotic behaviour of the solution of (1.2) near the extinction time is usually similar to the asymptotic behaviour of the self-similar solution of (1.1), in order to understand the behaviour of the solution of (1.2) near the extinction time we will first study various properties of the self-similar solutions of (1.1) in this paper.

Let $n \ge 3$, $0 < m < \frac{n-2}{n}$, $\rho_1 > 0$, $\beta \ge \frac{m\rho_1}{n-2-nm}$ and $\alpha = \frac{2\beta+\rho_1}{1-m}$. For any $\lambda > 0$, by Theorem 1.1 of [13] there exists a unique radially symmetric solution v_λ of the equation

$$\Delta v^m + \alpha v + \beta x \cdot \nabla v = 0, v > 0, \tag{1.3}$$

in $\mathbb{R}^n$ that satisfies $v_\lambda(0) = \lambda$. By [15], v_λ satisfies

$$\lim_{r\to\infty} r^2 v_\lambda(r)^{1-m} = \frac{2m(n-2-nm)}{(1-m)\rho_1}. \tag{1.4}$$

Note that when $\rho_1 = 1$, the function

$$\psi_\lambda(x,t) = (T-t)^\alpha v_\lambda((T-t)^\beta x) \tag{1.5}$$

is a solution of (1.1) in $\mathbb{R}^n \times (0,T)$ for any $T > 0$. On the other hand if $\rho_1 = 1$, $m = \frac{n-2}{n+2}$, and $n \ge 3$, then the metric

$$g = \left(\left(\frac{n-1}{m}\right)^{\frac{1}{1-m}} v_\lambda\right)^{\frac{4}{n+2}} dx^2 \tag{1.6}$$

on $\mathbb{R}^n$ is a Yamabe shrinking soliton [8]. Conversely as proved by P. Daskalopoulos and N. Sesum [8] any Yamabe shrinking soliton on complete locally conformally flat manifold with positive sectional curvature is of the form (1.6) where v_λ is a solution of (1.3) in $\mathbb{R}^n$ for some $\alpha = \frac{2\beta+1}{1-m}$ with $v_\lambda(0) = \lambda$ for some constant $\lambda > 0$.

Let $\beta_1 = \frac{\rho_1}{n-2-nm}$,

$$\beta_0 = \begin{cases} \rho_1\sqrt{\dfrac{2(1-m)}{n-2-nm}} & \text{if } 0 < m \le \dfrac{n-2}{n+2} \\ \rho_1 \max\left(2\sqrt{\dfrac{2(1-m)}{n-2-nm}}, \dfrac{(n+2)m-(n-2)}{n-2-nm}\right) & \text{if } \dfrac{n-2}{n+2} < m < \dfrac{n-2}{n}, \end{cases}$$