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# Linking the boundary and exponential spectra via the restricted topology ☆

Sonja Mouton [a,*], Robin Harte [b]

[a] Department of Mathematical Sciences, Private Bag X1, Stellenbosch University, Matieland 7602, South Africa
[b] School of Mathematics, Trinity College, Dublin, Ireland

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ABSTRACT

We build a chain, based on subalgebras, connecting the boundary spectrum/exponential spectrum duality with the duality between the usual boundary and the connected hull.



## 1. Introduction

If $a$ is an element of a complex Banach algebra $A$ with unit 1, then the boundary spectrum $S_\partial(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \in \partial A^{-1}\}$ of $a$ (see [4]) is a compact set in $\mathbb{C}$ that lies between the usual topological boundary of the spectrum and the spectrum $\sigma(a)$ itself, i.e.

$$\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a).$$

It therefore seems natural to view $\partial\sigma(a)$, on the one hand, and $S_\partial(a)$, on the other hand, as the "thin" and "fat" boundaries of $\sigma(a)$, respectively.

In this paper we show that, using closed subalgebras $B$, it is possible to define a topology on $A$ (different from the norm-topology) in such a way that a whole range of "boundaries" can be obtained, with $B = \mathbb{C}$ giving the "thin" boundary and $B = A$ the "fat" boundary of $\sigma(a)$ — see Corollary 6.4.

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* Corresponding author.
*E-mail addresses:* smo@sun.ac.za (S. Mouton), rharte@maths.tcd.ie (R. Harte).

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From [3] we recall a certain duality between the boundary $\partial\sigma(a)$ and the connected hull $\eta\sigma(a)$ of the spectrum of $a$. Bearing in mind the fact that the exponential spectrum $\varepsilon(a)$ of $a$ lies between $\sigma(a)$ and $\eta\sigma(a)$, i.e.

$$\sigma(a) \subseteq \varepsilon(a) \subseteq \eta\sigma(a),$$

we also investigate the "connected hulls" accompanying these new "boundaries", and show that $B = A$ yields $\varepsilon(a)$, making $\varepsilon(a)$ the "little" connected hull of $\sigma(a)$. In addition it is shown that in trivial cases, the "big" connected hull of $\sigma(a)$ is $\eta\sigma(a)$, while otherwise it is actually the whole of $\mathbb{C}$ (see Corollary 6.9).

In the following section we provide all the relevant notation and terminology. In Section 3, given an additive topological group $A$ and a subgroup $B$ of $A$, we define a new topology on $A$, called the *$B$-topology* or the *restricted topology*, via the closure operation and show that, for elements and subsets of $A$ lying inside $B$, topological concepts in the $B$-topology coincide with those in the relative topology (see Theorem 3.4). We also obtain a number of basic properties relating to the $B$-topology and provide a number of examples.

In Section 4 we investigate the correspondence between subsets $H_\omega$ of $A$ and certain mappings $\omega$ from $A$ into $2^B$ relying on some special element $e$ of $B$. (The motivation for this is the relationship between the set of all invertible elements of a Banach algebra $A$ and the function that maps an element $a \in A$ onto its spectrum, where $B = \mathbb{C}$ and $e = 1$.) Section 5 is devoted to finding the relationships between the restricted boundaries and between the restricted connected hulls of $\omega$ and $A \backslash H_\omega$, with the main results contained in Theorems 5.2 and (its partial analogue) 5.13.

Finally, in Section 6, we return to Banach algebras. Using the concepts of the restricted boundary and the restricted connected hull of the set of all non-invertible elements of a Banach algebra $A$, we define a range of "boundaries" and "connected hulls" contained in $\mathbb{C}$ (see (6.1) and (6.6)) and thereby arrive at our main results Theorems 6.2 and 6.7. In conclusion, we show that certain known results about the boundary spectrum can be generalised by using our new "boundary" concept (see, in particular, Theorem 6.15).

## 2. Preliminaries

Let $X$ be a topological space and let $t$ be an element of $X$. Then we denote the set of all neighbourhoods of $t$ in $X$ by $\mathrm{Nbd}_X(t)$. In addition, if $K$ is a subset of $X$, then the (topological) closure, interior and boundary of $K$ in $X$ will be denoted by $\mathrm{cl}_X(K)$, $\mathrm{Int}_X(K)$ and $\partial_X(K)$, respectively.

In some sense dual to the topological boundary is the connected hull. If $t \notin K$, then the *connected hull* $\eta_t(K)$ *relative to* $t$ of $K$ is defined by

$$X \backslash \eta_t(K) = \mathrm{Comp}_X(t, X \backslash K),$$

where $\mathrm{Comp}_X(t, H) \subseteq X$ is the *(connected) component* of $t$ in $H$. When in particular $X$ is a normed linear space and $K$ is a bounded set in $X$, we shall also write $\eta K := \eta_\infty K$ for the complement of the unique unbounded component of $X \backslash K$. If instead $X$ is a topological ring with identity $e$, and $e \notin S \subseteq A$, we shall prefer $\eta S := \eta_e S$.

If, in turn, $A$ is a complex Banach algebra with unit 1, then the set of all invertible elements will be indicated by $A^{-1}$ and elements of the form $\lambda 1$ in $A$ will be denoted by $\lambda$. If $a$ is an element of $A$, then the *spectrum* $\{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\}$ of $a$ in $A$ will be denoted by $\sigma(a)$ (or by $\sigma_A(a)$, if necessary to avoid confusion). Referring to the notation given above, we will use the symbols $\partial\sigma(a) := \partial_{\mathbb{C}}\sigma(a)$ and (as already mentioned) $\eta\sigma(a) := \eta_\infty\sigma(a)$ for the boundary and the connected hull, respectively, of $\sigma(a)$. It is well known that if $B$ is a closed subalgebra of $A$ containing 1, then

$$\sigma_A(a) \subseteq \sigma_B(a) \quad \text{and} \quad \partial\sigma_B(a) \subseteq \partial\sigma_A(a) \tag{2.1}$$

(see, for instance, [1], Theorem 3.2.13).