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Angular equivalence of normed spaces

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ABSTRACT

Angular equivalence is introduced and shown to be an equivalence relation among the norms on a fixed real vector space. It is a finer notion than the usual (topological) notion of norm equivalence. Angularly equivalent norms share certain geometric properties: A norm that is angularly equivalent to a uniformly convex norm is itself uniformly convex. The same is true for strict convexity. Extreme points of the unit balls of angularly equivalent norms occur on the same rays, and if one unit ball is a polyhedron so is the other. Among norms arising from inner products, two norms are angularly equivalent if and only if they are topological equivalent. But, unlike topological equivalence, angular equivalence is able to distinguish between different norms on a finite-dimensional space. In particular, no two ℓ^p norms on $\mathbb{R}^n$ are angularly equivalent.

1. Introduction and definition

Two norms on a real vector space are equivalent if both give rise to the same notion of convergence. A wide variety of functional analysis results concern only the topology a norm generates, not the specific values taken by a given norm. In such a setting, choosing the most convenient one among several, or many, topologically equivalent norms can clarify arguments and simplify proofs.

In other situations, specific properties of individual norms are central to the theory. For example, uniform convexity is an important property of a norm that may not be shared by a topologically equivalent norm.

It is our object here to introduce a finer equivalence of norms, one that preserves certain properties of the norm that simple topological equivalence does not. The idea is straightforward; two norms are *angularly equivalent* if, over all pairs of non-zero vectors, the angle between the pair, determined by one norm, is comparable to the angle between the same pair, determined by the other norm. This will be made precise shortly.

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Although the theory of angles in normed spaces cannot match the elegance of its counterpart for inner product spaces, it serves here to give us a means of defining an equivalence of norms that compares vectors two by two rather than one at a time, as topological equivalence does. Thus, angular equivalence emerges as a kind of "second order" equivalence compared to the "first order" topological equivalence.

Definition 1. Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a real vector space X are *topologically equivalent* provided there exist positive constants m, M such that for all $x, y \in X$,

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1. \tag{1.1}$$

Very little is needed to give the definition of angular equivalence besides an accessible concept of angle in normed spaces. We define an angle based on the g-functional, introduced in [6] and studied in [2, Chapter 4] and references there. It is closely connected to smoothness and convexity properties of the unit ball and its simple definition permits straightforward calculations.

Fix vectors x and y in a real vector space X, with norm $\|\cdot\|$. A few applications of the triangle inequality are enough to show that $\frac{1}{t}(\|x+ty\| - \|x\|)$ is a non-decreasing function of t taking $(-\infty, 0) \cup (0, \infty)$ into $[-\|y\|, \|y\|]$. It follows that both

$$g^+(x, y) = \|x\| \lim_{t \to 0+} \tfrac{1}{t}(\|x+ty\| - \|x\|) \quad \text{and} \quad g^-(x, y) = \|x\| \lim_{t \to 0-} \tfrac{1}{t}(\|x+ty\| - \|x\|)$$

exist, and satisfy

$$-\|y\| \leq \|x\| - \|x-y\| \leq \frac{g^-(x, y)}{\|x\|} \leq \frac{g^+(x, y)}{\|x\|} \leq \|x+y\| - \|x\| \leq \|y\|. \tag{1.2}$$

Definition 2. Suppose $\|\cdot\|$ is a norm on a real vector space X. The *g-functional* relative to $\|\cdot\|$ is the map $g : X \times X \to [0, \infty)$ given by, $g = \frac{1}{2}(g^- + g^+)$. If x and y are non-zero vectors in X, the *norm angle*[3] from x to y is $\theta(x, y)$, defined by $0 \leq \theta \leq \pi$ and

$$\cos\theta(x, y) = \frac{g(x, y)}{\|x\|\|y\|}.$$

We will refrain from referring to the norm angle "between" x and y, since the norm angle from x to y may not coincide with the norm angle from y to x. If the norm in X arises from an inner product, it is easy to see that norm angles agree with angles defined by the inner product. To see that $\theta(x, y)$ does not depend on the lengths of x and y, make the substitution $s = bt/a$ in the defining limits to get $g(ax, by) = abg(x, y)$ whenever $a, b > 0$. A little extra care shows that this equation holds for any $a, b \in \mathbb{R}$ so, in particular, $g(x, -y) = -g(x, y)$.

Definition 3. Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a real vector space X are *angularly equivalent* provided there exists a constant C such that for all non-zero $x, y \in X$,

$$\tan(\theta_2(x, y)/2) \leq C \tan(\theta_1(x, y)/2). \tag{1.3}$$

Here $\theta_1(x, y)$ and $\theta_2(x, y)$ are the norm angles from x to y relative to $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Also, $\tan(\pi/2)$ is taken to be $+\infty$.

[3] The term "g-angle" is in use, coined by Pavle Miličić in [7] for an angle based on a symmetrized g-functional.