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Radial continuous valuations on star bodies

Pedro Tradacete [a,1], Ignacio Villanueva [b,*,2]

[a] *Mathematics Department, Universidad Carlos III de Madrid, 28911 Leganés (Madrid), Spain*
[b] *Departamento de Análisis Matemático, Instituto de Matemática Interdisciplinar IMI, Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid 28040, Spain*

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ABSTRACT

We show that a radial continuous valuation defined on the n-dimensional star bodies extends uniquely to a continuous valuation on the n-dimensional bounded star sets. Moreover, we provide an integral representation of every such valuation, in terms of the radial function, which is valid on the dense subset of the simple Borel star sets. Along the way, we also show that every radial continuous valuation defined on the n-dimensional star bodies can be decomposed as a sum $V = V^+ - V^-$, where both V^+ and V^- are positive radial continuous valuations.

1. Introduction

This note continues the study of valuations on star bodies started in [18]. A valuation is a function V, defined on a class of sets, with the property that

$$V(A \cup B) + V(A \cap B) = V(A) + V(B).$$

As a generalization of the notion of measure, valuations have become a relevant area of study in convex geometry. In fact, this notion played a critical role in M. Dehn's solution to Hilbert's third problem, asking whether an elementary definition for volume of polytopes was possible. See, for instance, [15,16] and the references there included for a broad vision of the field.

Valuations on convex bodies belong to the Brunn–Minkowski theory. This theory has been extended in several important ways, and in particular, to the dual Brunn–Minkowski theory, where convex bodies, Minkowski addition and Hausdorff metric are replaced by star bodies, radial addition and radial metric, respectively. The dual Brunn–Minkowski theory, initiated in [17], has been broadly developed and successfully

* Corresponding author.
E-mail addresses: ptradace@math.uc3m.es (P. Tradacete), ignaciov@mat.ucm.es (I. Villanueva).

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applied to several areas, such as integral geometry, local theory of Banach spaces and geometric tomography (see [5,10] for these and other applications). In particular, it played a key role in the solution of the Busemann–Petty problem [9,11,19].

D. A. Klain initiated in [13,14] the study of rotationally invariant valuations on a certain class of star sets, namely those whose radial function is n-th power integrable.

In [18], the second named author started the study of valuations on star bodies, characterizing positive rotation invariant valuations as those described by a certain integral representation.

The assumption of rotational invariance strongly simplifies the analysis in [18]. In this note, we drop that assumption and study continuous valuations on star bodies without further restrictions.

The main question in this context is whether radial continuous valuations in general admit an integral representation in the spirit of the representation valid for rotation invariant valuations. Such a representation would provide a detailed understanding of valuations. We do not fully answer this question, but we do give a partially positive answer which is probably already sufficient for many applications.

Our main result states that every radial continuous valuation can be extended to a continuous valuation on the bounded star sets, and this extension provides an integral representation of the valuation on the star sets with a simple Borel radial function. Note that the star sets with simple Borel radial function are dense, with the radial metric, in the space of bounded star sets.

For the sake of clarity, we split the result in two statements.

Theorem 1.1. *Let $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ be a radial continuous valuation on the n-dimensional star bodies $\mathcal{S}_0^n$. Then, there exists a unique radial continuous extension of V to a valuation $\overline{V} : \mathcal{S}_b^n \longrightarrow \mathbb{R}$ on the bounded Borel star sets of $\mathbb{R}^n$.*

Theorem 1.2. *Let $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ be a radial continuous valuation, and let $\overline{V}$ be its extension mentioned in Theorem 1.1. Then, there exists a Borel measure μ on S^{n-1} and a function $K : \mathbb{R}^+ \times S^{n-1} \to \mathbb{R}$ such that, for every star body L whose radial function ρ_L is a simple function, we have*

$$\overline{V}(L) = \int_{S^{n-1}} K(\rho_L(t), t)d\mu(t).$$

The main technical difficulties arise in the proof of Theorem 1.1. In the rotation invariant case, the uniqueness of the Lebesgue measure among normalized rotation invariant measures on the unit sphere of $\mathbb{R}^n$ greatly simplified the study of the problem. In the general case, we do not have an equivalent result. Mimicking the techniques of [18], it is not too difficult to define a new valuation on the simple star sets. Difficulties arise when trying to extend this valuation to the bounded star sets, in order to check that it coincides with the original one. We do not know whether radial continuous valuations are uniformly continuous on bounded sets. For that reason, we do not know a priori that the valuation defined on the simple star sets preserves Cauchy sequences and can, therefore, be extended to its completion. We need to go through elaborate reasonings, especially in Section 6, to overcome this problem.

To prove Theorems 1.1 and 1.2, we also need an independent auxiliary result: a Jordan-like decomposition which will probably find applications elsewhere. We show that every continuous valuation $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ on the n-dimensional star bodies can be decomposed as the difference of two positive continuous valuations. With this structural result at hand, the study of continuous valuations on star bodies reduces to the simpler case of positive continuous valuations.

Theorem 1.3. *Let $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ be a radial continuous valuation on the n-dimensional star bodies $\mathcal{S}_0^n$ such that $V(\{0\}) = 0$. Then, there exist two radial continuous valuations $V^+, V^- : \mathcal{S}_0^n \longrightarrow \mathbb{R}_+$ such that $V^+(\{0\}) = V^-(\{0\}) = 0$ and such that*