Doctopic: Partial Differential Equations ARTICLE IN PRESS YJMAA:21083

J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Absence of nonnegative solutions to the system of differential inequalities on manifolds

Yuhua Sun

School of Mathematical Sciences and LPMC, Nankai University, 300071 Tianjin, PR China

ARTICLE INFO

Article history:
Received 28 July 2016
Available online xxxx
Submitted by H.R. Parks

Keywords:
System of differential inequalities
Critical exponent
Riemannian manifold
Volume growth

ABSTRACT

We mainly investigate the nonexistence of non-negative solution to the system of differential inequalities

$$\begin{cases} \Delta u + u^\tau v^m \le 0, \\ \Delta v + v^\eta u^n \le 0, \end{cases} \tag{1}$$

on a complete connected non-compact Riemannian manifold, where $\tau, \eta \geq 0$, $m, n > 0$ are given parameters satisfying $\tau + m = \eta + n = \sigma > 1$. We prove that, for some reference point x_0 if

$$\mu(B(x_0, r)) \le C r^{\frac{2\sigma}{\sigma-1}} (\ln r)^{\frac{1}{\sigma-1}}, \tag{2}$$

holds for all large enough r. Then (1) admits only trivial solution. Here $B(x_0, r)$ is a geodesic ball. We also show the sharpness of the volume growth condition (2).

1. Introduction

In this paper, we mainly investigate the nonexistence of non-negative solution to the system of differential inequalities

$$\begin{cases} \Delta u + u^\tau v^m \le 0, \\ \Delta v + v^\eta u^n \le 0, \end{cases} \tag{1.1}$$

on a geodesically complete connected non-compact Riemannian manifold (M, d, μ). Here $d(x, y)$ is the geodesic distance, and μ is the Riemannian measure, $\tau, \eta \geq 0, m, n > 0$ are given parameters satisfying $\tau + m = \eta + n = \sigma > 1$. Δ is the Laplace–Beltrami operator on manifold.

E-mail address: sunyuhua@nankai.edu.cn.

http://dx.doi.org/10.1016/j.jmaa.2017.01.057
0022-247X/© 2017 Published by Elsevier Inc.

Please cite this article in press as: Y. Sun, Absence of nonnegative solutions to the system of differential inequalities on manifolds, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.01.057

The goal of the paper is to provide a simple geometric condition on M ensuring that there exists no nontrivial solution to (1.1).

The study of the nonexistence of the solution always attracted considerable mathematical interests. For the scalar case in $\mathbb{R}^N$, namely

$$\Delta u + u^\sigma \leq 0. \tag{1.2}$$

It is well known that (1.2) admits only trivial solution $u \equiv 0$ if and only if $\sigma \leq \frac{N}{N-2}$ for $N \geq 3$. A number of the generalizations to a more general equation and differential inequalities have been obtained. Recall the famous Lane–Emden system in $\mathbb{R}^N$

$$\begin{cases} \Delta u + v^{\sigma_1} = 0, \\ \Delta v + u^{\sigma_2} = 0. \end{cases} \tag{1.3}$$

Here $\sigma_1, \sigma_2 > 0$. Recall the famous Lane–Emden conjecture: if

$$\frac{1}{\sigma_1 + 1} + \frac{1}{\sigma_2 + 1} > 1 - \frac{2}{N},$$

then system (1.3) admits no positive classical solutions. In [4], by applying the powerful Alexandrov–Serrin moving plane method and blow-up arguments, Figueiredo and Felmer proved that

(i) The system (1.3) admits no positive solution provided that if

$$0 < \sigma_1, \sigma_2 \leq \frac{N+2}{N-2}, \quad (\sigma_1, \sigma_2) \neq (\frac{N+2}{N-2}, \frac{N+2}{N-2}).$$

(ii) If $\sigma_1 = \sigma_2 = \frac{N+2}{N-2}$, the solution (u, v) to (1.3) is radially symmetric with respect to some point of $\mathbb{R}^N$.

However there is still little known about the Lane–Emden conjecture. The conjecture is affirmed only in the low dimension case: when $N = 3$, the conjecture was proved by Serrin and Zou [21] under the case of polynomially bounded solutions, and then the assumption was completely removed by Poláčik, Quittner and Souplet [19]. When $N = 4$, this was established by Souplet in [22]. But for general dimensions, so far the conjecture is still unsolved. In [15], Mitidieri and Pohozaev proved (1.3) has no positive classical supersolutions in $\mathbb{R}^N$ with $N \geq 3$. They gave a critical sharp condition concerning the nonexistence of the solution, namely, if

$$\max\left\{\frac{2\sigma_1(\sigma_2 + 1)}{\sigma_1\sigma_2 - 1}, \frac{2\sigma_2(\sigma_1 + 1)}{\sigma_1\sigma_2 - 1}\right\} \geq N, \tag{1.4}$$

then there exists only trivial nonnegative supersolution (u, v) to (1.3). The sharpness of the above condition can be showed by a very simple example: if (1.4) fails, then (1.3) admits a positive supersolution by choosing small enough ϵ

$$u(x) = \epsilon(1 + |x|^2)^{\frac{\sigma_1+1}{1-\sigma_1\sigma_2}}, \quad v(x) = \epsilon(1 + |x|^2)^{\frac{\sigma_2+1}{1-\sigma_1\sigma_2}}.$$

They developed a well used tool named test function techniques (or nonlinear capacity approach). Their approach can be systematically applicable to many types of differential inequalities especially in Euclidean space, such as quasilinear elliptic and even parabolic differential inequalities. Let us recommend a series of papers [1,2,10,14,16–18] for a more comprehensive description.

Please cite this article in press as: Y. Sun, Absence of nonnegative solutions to the system of differential inequalities on manifolds, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.01.057