J. Math. Anal. Appl. 450 (2017) 954–968

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Differentiability of a two-parameter family of self-affine functions

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ARTICLE INFO

Article history:
Received 3 August 2016
Available online 26 January 2017
Submitted by B. Bongiorno

Keywords:
Continuous nowhere differentiable function
Infinite derivative
Beta-expansion
Hausdorff dimension
Komornik–Loreti constant
Thue–Morse sequence

ABSTRACT

This paper highlights an unexpected connection between expansions of real numbers to noninteger bases (so-called β-*expansions*) and the infinite derivatives of a class of self-affine functions. Precisely, we extend Okamoto's function (itself a generalization of the well-known functions of Perkins and Katsuura) to a two-parameter family $\{F_{N,a} : N \in \mathbb{N}, 1/(N+1) < a < 1\}$. We first show that for each x, $F'_{N,a}(x)$ is either 0, $\pm\infty$, or undefined. We then extend Okamoto's theorem by proving that for each N, depending on the value of a relative to a pair of thresholds, the set $\{x : F'_{N,a}(x) = 0\}$ is either empty, uncountable but Lebesgue null, or of full Lebesgue measure. We compute its Hausdorff dimension in the second case. The second result is a characterization of the set $\mathcal{D}_\infty(a) := \{x : F'_{N,a}(x) = \pm\infty\}$, which enables us to closely relate this set to the set of points which have a unique expansion in the (typically noninteger) base $\beta = 1/a$. Recent advances in the theory of β-expansions are then used to determine the cardinality and Hausdorff dimension of $\mathcal{D}_\infty(a)$, which depend qualitatively on the value of a relative to a second pair of thresholds.

1. Introduction

The aim of this paper is to investigate the differentiability of a two-parameter family of self-affine functions, constructed as follows. Fix a positive integer N and a real parameter a satisfying $1/(N+1) < a < 1$, and let b be the number such that $(N+1)a - Nb = 1$. Note that $0 < b < a$. Let $x_i := i/(2N+1)$, $i = 0, 1, \ldots, 2N+1$, and for $j = 0, 1, \ldots, N$, put $y_{2j} := j(a-b)$, and $y_{2j+1} := (j+1)a - jb$. Now set $f_0(x) := x$, and for $n = 1, 2, \ldots$, define f_n recursively on each interval $[x_i, x_{i+1}]$ ($i = 0, 1, \ldots, 2N$) by

$$f_n(x) := y_i + (y_{i+1} - y_i) f_{n-1}\big((2N+1)(x - x_i)\big), \qquad x_i \le x \le x_{i+1}. \tag{1}$$

Each f_n is a continuous, piecewise linear function from the interval $[0,1]$ onto itself, and it is easy to see that the sequence (f_n) converges uniformly to a limit function which we denote by $F_{N,a}$. This function $F_{N,a}$ is continuous and maps $[0,1]$ onto itself. It may be viewed as the self-affine function "generated" by

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http://dx.doi.org/10.1016/j.jmaa.2017.01.073
0022-247X/

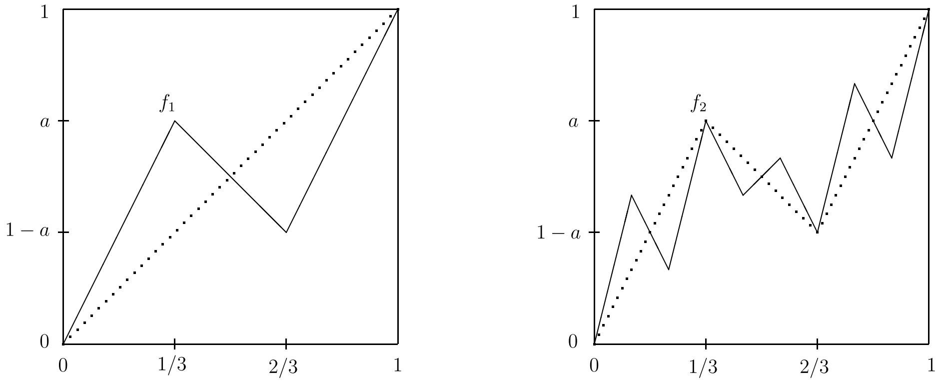

Fig. 1. The first two steps in the construction of $F_{1,a}$.

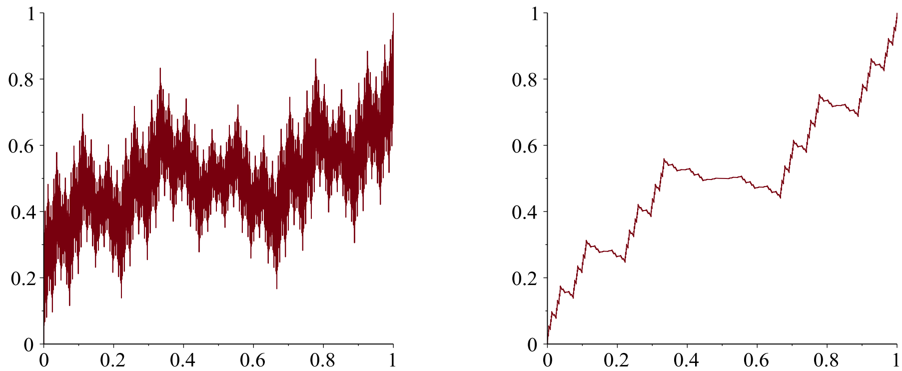

Fig. 2. Graph of $F_{1,a}$ for $a = 5/6$ (Perkins' function; left) and $a = \hat{a}_\infty \doteq .5598$ (right). See Section 2.2 for the definition and significance of $\hat{a}_\infty$.

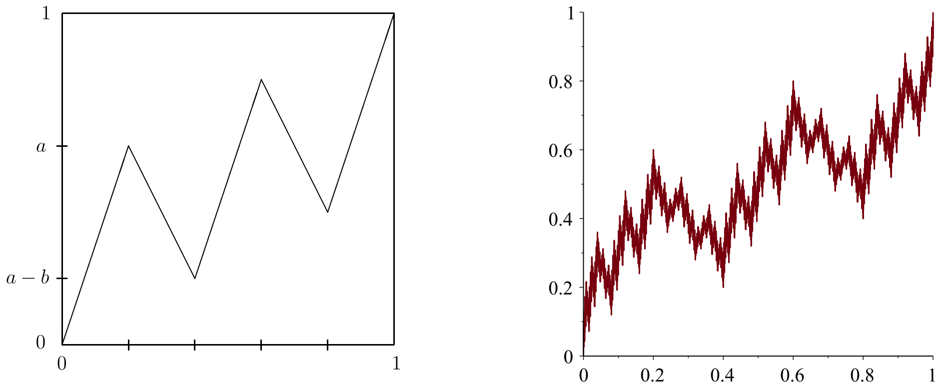

Fig. 3. The generating pattern and graph of $F_{2,a}$, shown here for $a = 0.6$.

the piecewise linear function with interpolation points (x_i, y_i), $i = 0, 1, \ldots, 2N + 1$. When $N = 1$ we have Okamoto's family of self-affine functions [23], which includes Perkins' function [25] for $a = 5/6$ and the Katsuura function [14] for $a = 2/3$; see also Bourbaki [4]. Fig. 1 illustrates the above construction for $N = 1$; graphs of $F_{1,a}$ for two values of a are shown in Fig. 2; and Fig. 3 illustrates the case $N = 2$.

The restriction $a > 1/(N + 1)$ is not necessary; when $a = 1/(N + 1)$ we have $b = 0$ and $F_{N,a}$ is a generalized Cantor function. When $0 < a < 1/(N + 1)$, we have $b < 0$ and $F_{N,a}$ is strictly increasing and singular. Since the differentiability of such functions has been well-studied (e.g. [6–9,12,15,22,27]), we will focus exclusively on the case $a > 1/(N + 1)$, when $F_{N,a}$ is of unbounded variation. However, see [1] for a detailed analysis of the case $N = 1$, $a \leq 1/2$.