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Coherent pairs of measures and Markov–Bernstein inequalities

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ABSTRACT

All the coherent pairs of measures associated to linear functionals c_0 and c_1, introduced by Iserles et al. in 1991, have been given by Meijer in 1997. There exist seven kinds of coherent pairs. All these cases are explored in order to give three term recurrence relations satisfied by polynomials. The smallest zero $\mu_{1,n}$ of each of them of degree n has a link with the Markov–Bernstein constant M_n appearing in the following Markov–Bernstein inequalities:

$$c_1((p')^2) \leq M_n^2 c_0(p^2), \quad \forall p \in \mathcal{P}_n,$$

where $M_n = \frac{1}{\sqrt{\mu_{1,n}}}$. The seven kinds of three term recurrence relations are given. In the case where $c_0 = e^{-x}dx + \delta(0)$ and $c_1 = e^{-x}dx$, explicit upper and lower bounds are given for $\mu_{1,n}$, and the asymptotic behavior of the corresponding Markov–Bernstein constant is stated. Except in a part of one case, $\lim_{n\to\infty} \mu_{1,n} = 0$ is proved in all the cases.

1. Introduction

The Markov–Bernstein inequalities in L^2 norm that imply two measures, have the following form

$$c_1((p')^2) \leq M_n^2 c_0(p^2), \quad \forall p \in \mathcal{P}_n,$$

where c_0 and c_1 denote the two linear functionals associated to the both measures. p' is the derivative of p. $\mathcal{P}_n$ is the vector space of real polynomials in one variable of degree at most n. M_n is called Markov–Bernstein constant. These inequalities are always related to an eigenvalue problem of a positive definite symmetric matrix (see [13,4] for a general presentation). For any measures this matrix generally is full. But for classical measures (Hermite, Laguerre–Sonin, Jacobi) this matrix has a particular form. It is diagonal for all these

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measures (see [3]) if $c_0 = c_1 = \int_{-\infty}^{+\infty} .e^{-x^2} dx$ in the Hermite case, if $c_0 = c^\alpha = \int_0^{+\infty} .x^\alpha e^{-x} dx$ with $\alpha > -1$ and $c_1 = c^{\alpha+1}$ in the Laguerre–Sonin case, if $c_0 = c^{(\alpha,\beta)} = \int_{-1}^{+1} .(1-x)^\alpha (1+x)^\beta dx$ with $\alpha > -1$ and $\beta > -1$, and $c_1 = c^{(\alpha+1,\beta+1)}$ in the Jacobi case. It is tridiagonal (see [13,4]) if $c_0 = c_1 = c^\alpha$ in the Laguerre–Sonin case. If $c_0 = c_1 = c^{(\alpha,\alpha)}$ that is the Gegenbauer case, two tridiagonal matrices are obtained (see [13,4]). It is a five-diagonal matrix if $c_0 = c_1 = c^{(\alpha,\beta)}$ in the Jacobi case (see [4,7]). In the discrete case, if c_0 and c_1 are identical and correspond to the Meixner measure, the matrix is also tridiagonal (see [5,1]). When the matrix is tridiagonal, its characteristic polynomial satisfies a three term recurrence relation. To find the Markov–Bernstein constant is equivalent to find the smallest zero $\mu_{1,n}$ of the polynomial of degree n satisfying this three term recurrence relation, and $M_n = 1/\sqrt{\mu_{1,n}}$. The characteristic polynomial in the Jacobi case $c_0 = c_1 = c^{(\alpha,\beta)}$ provides a six-term recurrence relation (see [7]). Thus, to obtain a tridiagonal matrix is exceptional. There exists another case for which a tridiagonal matrix is obtained: it is the one of coherent pairs (c_0, c_1).

The notion of coherent pairs of measures was introduced for first time by Iserles et al. [8] in 1991. These measures are defined as follows

Definition 1.1. Let c_0 and c_1 be two quasi-definite linear functionals. Let $\{P_n\}_{n\geq 0}$ (resp. $\{T_n\}_{n\geq 0}$) be the sequence of monic orthogonal polynomials with respect to c_0 (resp. c_1). (c_0, c_1) is called coherent pair if and only if there exists a sequence $\{\sigma_n\}_{n\geq 1}$, $\sigma_n \in \mathbb{R}$, $\sigma_n \neq 0$, such that

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \quad \forall n \geq 1. \tag{1}$$

All the kinds of coherent pairs (c_0, c_1) were described by Meijer [11] in 1997. They correspond to the relations (3.12) to (3.18) of his paper. One of both functionals is classical (Laguerre–Sonin or Jacobi). The seven cases contained in [11] are given below.

1. **Laguerre case:** $\Omega =]0, +\infty[$
   (a) c_0 corresponds to the measure $(x - \xi)x^{\alpha-1}e^{-x}dx$ with $\alpha > 0$ and $\xi < 0$.
   c_1 corresponds to the Laguerre–Sonin measure $x^\alpha e^{-x}dx$.
   (b) c_0 corresponds to the measure $e^{-x}dx + M\delta(0)$ with $M \geq 0$. δ is the Dirac measure.
   c_1 corresponds to the Laguerre measure $e^{-x}dx$.
   (c) c_0 corresponds to the Laguerre–Sonin measure $x^\alpha e^{-x}dx$ with $\alpha > -1$.
   c_1 corresponds to the measure $\frac{x^{\alpha+1}}{x-\xi}e^{-x}dx + M\delta(\xi)$ with $\xi \leq 0$ and $M \geq 0$.
2. **Jacobi case:** $\Omega =]-1, +1[$
   (a) c_0 corresponds to the measure $\mid x - \xi \mid (1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ with $\alpha > 0$, $\beta > 0$ and $\mid \xi \mid > 1$.
   c_1 corresponds to the Jacobi measure $(1-x)^\alpha(1+x)^\beta dx$.
   (b) c_0 corresponds to the measure $(1+x)^{\beta-1}dx + M\delta(1)$ with $\beta > 0$ and $M \geq 0$.
   c_1 corresponds to the Jacobi measure $(1+x)^\beta dx$.
   (c) c_0 corresponds to the measure $(1-x)^{\alpha-1}dx + M\delta(-1)$ with $\alpha > 0$ and $M \geq 0$.
   c_1 corresponds to the Jacobi measure $(1-x)^\alpha dx$.
   (d) c_0 corresponds to the Jacobi measure $(1-x)^\alpha(1+x)^\beta dx$ with $\alpha > -1$ and $\beta > -1$.
   c_1 corresponds to the measure $\frac{1}{|x-\xi|}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + M\delta(\xi)$ with $\mid \xi \mid \geq 1$ and $M \geq 0$.

In every previous cases one measure depends on one parameter ξ or M, or both ξ and M. Our aim is to fix these parameters, and therefore to fix the different measures. From these measures we want to give, in an explicit form only depending on the fixed parameters, the two sequences of polynomials $\{P_n\}_{n\geq 0}$ and $\{T_n\}_{n\geq 0}$ as well as their L^2 norms, and the sequence $\{\sigma_n\}_{n\geq 1}$. After that part we want to obtain, also in an explicit form only depending on the fixed parameters, the three term recurrence relation linked with the