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Approximation of discontinuous signals by sampling Kantorovich series

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ABSTRACT

In this paper, the behavior of the sampling Kantorovich operators has been studied, when discontinuous functions (signals) are considered in the above sampling series. Moreover, the rate of approximation for the family of the above operators is estimated, when uniformly continuous and bounded signals are considered. Finally, several examples of (duration-limited) kernels which satisfy the assumptions of the present theory have been provided, and also the problem of the linear prediction by sampling values from the past is analyzed.

1. Introduction

The sampling Kantorovich operators $S_w f$ studied in this paper (see Section 2), first introduced in [7] in a one-dimensional setting, arise as a development of the generalized sampling operators.

More precisely, the generalized sampling operators are defined by:

$$\text{(I)} \qquad (G_w f)(t) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wt-k), \quad t\in\mathbb{R},$$

where the function $\chi:\mathbb{R}\to\mathbb{R}$ is a suitable kernel, satisfying certain assumptions. The above operators $G_w f$ were introduced in the 1980s and have been widely studied under various aspects, see e.g. [13,15,35,20,37,31,32,21,38]. In practice, the generalized sampling operators represent an approximate version of the classical Whittaker–Kotelnikov–Shannon sampling theorem which uses an L^1-kernel, see e.g. [36,11,10,28–30,6,3,2,8].

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On the other hand, in the sampling Kantorovich operators, in place of the sample values $f(k/w)$ we have mean values of the signal f, of the form $w\int_{k/w}^{(k+1)/w} f(u)\,du$, $k \in \mathbb{Z}$, and $w > 0$; it turns out that these operators reduce "time-jitter" errors, what is very useful in signal processing.

Since the behavior of the sampling Kantorovich series has been studied pointwise only at the continuity point of a given signal f, it still remains an open problem to study their behavior at the discontinuity points of f. Note that, even if the above averages make the sampling Kantorovich operators more regular, for what concerns the convergence at the discontinuity points of f, due to technical reasons, the situation becomes rather delicate.

The above issue is strictly related to the applications of the operators $S_w f$ for image reconstruction and enhancement (see e.g. [24,19]), taking into account that images are typical examples of discontinuous signals.

The problem of approximating signals at the discontinuity points is one of the main topics investigated in the present paper. First of all, we establish a representation formula (see Lemma 2.5) for the sampling Kantorovich series evaluated at a certain fixed time t where the signal f has a jump discontinuity, exploiting an auxiliary function appropriately defined. Then, by using the above formula, we become able to obtain some necessary and sufficient conditions for the convergence of the family $(S_w f)_{w>0}$ to a suitable finite linear combination of $f(t+0)$ and $f(t-0)$, where $f(t+0)$ and $f(t-0)$ are respectively the right and left limit of f.

Another useful question here studied is the rate of approximation of the operators $S_w f$ (see also [4,25]). In particular, in [4] it is proved an estimate for the order of approximation involving the modulus of continuity of the function being approximated, requiring that the discrete absolute moment of order $\beta \geq 1$ of the kernel used to construct the operators is finite (i.e. $m_\beta(\chi) < +\infty$, with $\beta \geq 1$). Since in general examples of kernels for which the discrete moment $m_\beta(\chi) = +\infty$, for $\beta \geq 1$, can be given, in the present paper we achieve an estimate applicable to sampling Kantorovich operators based upon these kernels.

Finally, we study the problem of the linear prediction of signals from sample values taken only from the past. This problem is important in real-world case studies, when, in order to reconstruct a given signal at time t, one knows the sample values only in the past of the present time t.

In all the above mentioned problems, a crucial role is played by the kernels used to construct the sampling Kantorovich operators. For this reason, a detailed discussion concerning the kernels is given at the end of the paper together with several examples.

In conclusion, we recall that the sampling Kantorovich operators, besides applications to image processing, have been largely studied also from the theoretical point of view. For instance, a nonlinear version of $S_w f$ has been studied in [24] both in continuous and in Orlicz functions spaces; while a more general version of $S_w f$ has been considered in [39]. We recall that, Orlicz spaces are very general spaces including, among its various special cases, the L^p-spaces (see e.g. [34,33,5]).

2. Approximation of discontinuous signals

First of all, we introduce the family of discrete operators studied in this paper.

In what follows, a function $\chi : \mathbb{R} \to \mathbb{R}$ will be called a *kernel* if it satisfies the following conditions:

(χ1) $\chi \in L^1(\mathbb{R})$ and it is bounded in $[-1, 1]$;

(χ2) for every $u \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \chi(u-k) = 1; \tag{1}$$

(χ3) for some $\beta > 0$, the discrete absolute moment of order β are finite, i.e.,