J. Math. Anal. Appl. 450 (2017) 1104–1122

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stability in a scalar differential equation with multiple, distributed time delays

Sue Ann Campbell [a,*], Israel Ncube [b]

[a] *Department of Applied Mathematics and Centre for Theoretical Neuroscience, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada*
[b] *College of Engineering, Technology & Physical Sciences, Department of Mathematics, Alabama A & M University, 4900 Meridian Street N., Huntsville, AL 35762, USA*

A R T I C L E   I N F O

Article history:
Received 1 November 2016
Available online 27 January 2017
Submitted by J. Shi

Keywords:
Delay differential equations
Distributed delay
Stability boundaries

A B S T R A C T

We consider a linear scalar delay differential equation (DDE), consisting of two arbitrary distributed time delays. We formulate necessary conditions for stability of the trivial solution which are independent of the distributions. For the case of one discrete and one gamma distributed delay, we give an explicit description of the region of stability of the trivial solution and discuss how this depends on the model parameters.

1. Introduction

Distributed time delays arise in models for a variety of applications including population dynamics [12,14,18,20], blood cell dynamics [8,32], neuronal models [17,29], and coupled oscillators [5,6]. Although many of these models include only a single time delay, this often results from some simplification in the model set up. As an example, it is common to assume that all of the time delays are identical [17] or to neglect relatively smaller time delays [18]. The stability of equilibria in models with two discrete time delays has been studied extensively [7,15,24,33]. It has been shown that the presence of two time delays can lead to phenomena such as stability switching and the existence of codimension two bifurcation points [7,33]. In this article, we investigate such phenomena in a model with two distributed time delays. We focus our attention on the following scalar delay differential equation (DDE) with a linear decay:

$$\dot{x}(t) = -kx(t) + \alpha \int_0^\infty x(t-\tau) f_\alpha(\tau)\, d\tau + \beta \int_0^\infty x(t-\tau) f_\beta(\tau)\, d\tau \;, \tag{1}$$

* Corresponding author.
E-mail addresses: sacampbell@uwaterloo.ca (S.A. Campbell), Ncube.Israel@gmail.com (I. Ncube).

http://dx.doi.org/10.1016/j.jmaa.2017.01.060

where k, α, β are real numbers, $f_\alpha(T)$ and $f_\beta(T)$ are *arbitrary* distributions, satisfying

$$\int_0^\infty f_\alpha(s)ds = 1 = \int_0^\infty f_\beta(s)ds \;.$$

We note that (1) is a delay differential equation with infinite delay. Thus, the appropriate phase space is $C_{0,\rho}((-\infty,0],\mathbb{R})$, where ρ is a positive constant [13,16,19]. This is the Banach space of functions $\psi : (-\infty,0] \to \mathbb{R}$ such that $e^{\rho\,\theta}\psi(\theta)$ is continuous, and

$$\lim_{\theta\to\infty} e^{\rho\,\theta}\psi(\theta) = 0,$$

with norm $\|\psi\|_{\infty,\rho} = \sup_{\theta\le 0} e^{\rho\,\theta}\psi(\theta)$. In this space, we need the following additional conditions on the distributions:

$$\int_0^\infty e^{\rho\,s} f_\alpha(s)\,ds < \infty, \quad \int_0^\infty e^{\rho\,s} f_\beta(s)\,ds < \infty.$$

Since equation (1) is linear, it has a unique solution for any initial function $\phi \in C_{0,\rho}((-\infty,0],\mathbb{R})$ [13,16,19].

Stability with general distributions has been studied by some authors, but generally only with a single delay [4,8,27,30,31]. In particular, we note the work of Anderson [1,2] which studies stability properties of a linear, scalar differential equation with a single distributed time delay, in terms of the moments of the distribution.

Stability in the presence of multiple distributed delays has been studied in some models, generally by fixing the distributions to some specific form [3,25,26,21,28]. An exception is the work of Faria and Oliveira [14] which studies the global stability of equilibria in a class of Lotka–Volterra models with multiple distributed delays having finite maximum delay. They give conditions on the interaction coefficients of the system which guarantee asymptotic stability for any distribution.

Various specific time delay kernels have been used in the literature. The two most commonly used ones are the *weak* and the *strong* kernels (gamma distributions), given by $f(s) = re^{-rs}$ and $f(s) = r^2 se^{-rs}$, with $r > 0$, respectively. It is well-known (see [23] and [22], for instance) that the average time delays associated with the weak and the strong delay kernels are given by $T = \frac{1}{r}$ and $T = \frac{2}{r}$, respectively. Equation (1) would occur in the linearisation about an equilibrium point for the models of [7,15] and [33,26] if the discrete delays were replaced by distributed delays.

Making the change of variables $\widetilde{x} = x$, $\widetilde{t} = kt$, and defining new parameters by $\widetilde{\alpha} = \frac{\alpha}{k}$, $\widetilde{\beta} = \frac{\beta}{k}$, $T = k\tau$, and new distributions

$$g_\alpha(T) = \frac{1}{k} f_\alpha\left(\frac{T}{k}\right), \quad g_\beta(T) = \frac{1}{k} f_\beta\left(\frac{T}{k}\right),$$

we rescale (1) to get

$$\dot{x}(t) = -x(t) + \alpha \int_0^\infty x(t-T) g_\alpha(T)\,dT + \beta \int_0^\infty x(t-T) g_\beta(T)\,dT \;, \tag{2}$$

where, for notational tractability, we have dropped the tildes.

In this paper, we will investigate the stability of the trivial solution of (2) by direct analysis of the associated characteristic equation. The paper is organised as follows. In Section 2, we formulate some necessary distribution-independent conditions for stability of the trivial solution. In Section 3, we describe