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On the estimates of the $\mathcal{Z}$-eigenpair for an irreducible nonnegative tensor

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ABSTRACT

In this paper, we are concerned with the $\mathcal{Z}$-eigenpair of a tensor, in particular, an irreducible nonnegative tensor. Some new lower and upper bounds for the eigenvector and $\mathcal{Z}$-spectral radius of an irreducible (weakly symmetric) nonnegative tensors are provided. Our new bounds, which mostly generalize the ones presented in Li et al. (2015) [13], are closely related to the order of a tensor and proved to be tighter than those there. A new bound for $\mathcal{Z}_1$-eigenvalue of general tensors is specifically presented. Some examples are given to show the sharpness of our new bounds in contrast with the known ones, including the comparison results with the very recent research by other authors in the Appendix.

1. Introduction

Let $\mathbb{R}$ be the real field. An mth order n dimensional tensor $\mathcal{A}$ consists of n^m entries in $\mathbb{R}$, which is defined as follows:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \quad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \le i_1, i_2, \cdots, i_m \le n.$$

$\mathcal{A}$ is called *nonnegative (positive)* if $a_{i_1 \cdots i_m} \ge 0$ ($a_{i_1 \cdots i_m} > 0$). As usually, we denote the set of all mth order n dimensional tensors by $\mathbb{R}^{[m,n]}$ and the set of all nonnegative (positive) mth order n dimensional tensors by $\mathbb{R}_+^{[m,n]}$ ($\mathbb{R}_{++}^{[m,n]}$). A tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called *reducible* if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 \cdots i_m} = 0, \quad \forall i_1 \in I, \ \forall i_2, \ldots, i_m \notin I.$$

Otherwise, we call $\mathcal{A}$ *irreducible.*

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Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $x = (x_1, x_2, \cdots, x_n)^T$ be an n dimensional vector, real or complex. We define the n-dimension vectors

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1\le i\le n}$$

and $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \cdots, x_n^{m-1})^T$. A tensor $\mathcal{A}$ is called *weakly symmetric* if the associated homogeneous polynomial

$$f_{\mathcal{A}}(x) := x^T \mathcal{A}x^{m-1} = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2\cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

satisfies $\nabla f_{\mathcal{A}}(x) = m\mathcal{A}x^{m-1}$.

Qi [20] and Lim [15] independently introduced the following definitions.

Definition 1.1. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\mathbb{C}$ be the complex field. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue–eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

In particular, we call (λ, x) an $\mathcal{H}$-eigenpair if they are both real.

Definition 1.2. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\mathbb{C}$ be the complex field. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$ if they satisfy the equations

$$\begin{cases} \mathcal{A}x^{m-1} & = \lambda x, \\ x^T x & = 1. \end{cases}$$

We call (λ, x) a $\mathcal{Z}$-eigenpair if they are both real.

If a real pair (λ, x) satisfies

$$\mathcal{A}x^{m-1} = \lambda x, \qquad \|x\|_1 = 1,$$

then we call it a $\mathcal{Z}_1$-eigenpair [4]. In contrast to the $\mathcal{Z}_1$-eigenpair, $\mathcal{Z}$-eigenpair is sometimes called $\mathcal{Z}_2$-eigenpair. Both $\mathcal{Z}_1$-eigenpair and $\mathcal{Z}_2$-eigenpair are closely related. Chang and Zhang (Theorem 1.3 of [4]) have proved that (λ, x) is a $\mathcal{Z}_1$-eigenpair if and only if $(\frac{\lambda}{\|x\|_2^{m-2}}, \frac{x}{\|x\|_2})$ is a $\mathcal{Z}_2$-eigenpair.

Definition 1.3. [2] Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. We define the $\mathcal{Z}$-spectrum of $\mathcal{A}$, denoted by $\mathcal{Z}(\mathcal{A})$, to be the set of all $\mathcal{Z}$-eigenvalues of $\mathcal{A}$. Assume $\mathcal{Z}(\mathcal{A}) \neq \emptyset$, then the $\mathcal{Z}$-spectral radius of $\mathcal{A}$, denoted by $\varrho(\mathcal{A})$, is defined as

$$\varrho(\mathcal{A}) = \max\{|\lambda| | \lambda \in \mathcal{Z}(\mathcal{A})\}.$$

The set $\mathcal{Z}(\mathcal{A})$ may be an empty set, but the $\mathcal{Z}$-spectral radius $\varrho(\mathcal{A})$ of $\mathcal{A}$ is bounded as long as $\mathcal{Z}(\mathcal{A}) \neq \emptyset$, see [2].

The $\mathcal{Z}$-spectral radius $\varrho(\mathcal{A})$ of a nonnegative tensor $\mathcal{A}$ may not be itself a positive $\mathcal{Z}$-eigenvalue of $\mathcal{A}$ (see Example 3.4 in [2]), but the following Perron–Frobenius type theorem for the $\mathcal{Z}$-eigenvalue of nonnegative tensors was provided by K.C. Chang et al. [2].