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A note on stability of Mackey–Glass equations with two delays

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ABSTRACT

If the Mackey–Glass equation

$$\dot{x}(t) = r(t)\left[\frac{ax(h(t))}{1 + x^{\nu}(g(t))} - x(t)\right]$$

with $a > 1$ and $\nu > 0$ incorporates not one but two variable delays, some new phenomena arise: there may exist non-oscillatory about the positive equilibrium unstable solutions, the effect of possible absolute stability for certain a and ν disappears. We obtain sufficient conditions for local and global stability of the positive equilibrium and illustrate the stability tests, as well as new effects of two different delays, with examples.

1. Introduction

Many models of mathematical biology are described by a delay differential equation

$$\dot{x}(t) = \sum_{k=1}^{m} F_k(t, x(h_1(t)), \dots, x(h_l(t))) - G(t, x(t)), \tag{1.1}$$

where F_k, G are nonnegative continuous functions. Here the functions F_k describe production incorporating delay, and G corresponds to the instantaneous mortality. Positivity, boundedness and persistence for solutions of (1.1) were investigated in our recent paper [7]. Similar models with some applications were considered in [2,4,7,8,14,17].

For (1.1), usual assumptions are the following: the functions F_k are either monotone or unimodal, $G(t, u)$ is monotone increasing in u, there is only one delay involved in each of F_k, and a positive equilibrium

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is unique. However, it is possible to consider more general models, for example, the modified Nicholson blowflies equation

$$\dot{x}(t) = \sum_{k=1}^{m} a_k(t)x(h_k(t))e^{-\lambda_k x(g_k(t))} - b(t)x(t), \quad t \geq 0, \tag{1.2}$$

(in the standard model, $h_k \equiv g_k$) and the modified Mackey–Glass type equation

$$\dot{x}(t) = \sum_{k=1}^{m} \frac{a_k(t)x(h_k(t))}{1 + x^{n_k}(g_k(t))} - \left(b(t) - \frac{c(t)}{1 + x^n(t)}\right) x(t), \quad t \geq 0, \tag{1.3}$$

which in the case $h_k \equiv g_k$ and $c \equiv 0$ coincides with the usual case [9].

There are also many generalizations of Eqs. (1.1)–(1.3) to the case of distributed delays and integro-differential equations [6,11,12,16,18]. However, whatever complicated delay is involved, there is usually exactly one delay incorporated in each nonlinear function. This guarantees that the delay model, at least for small delays, inherits some properties of a nondelay system, for example, all nonoscillatory about the unique positive equilibrium solutions converge to this equilibrium, and for small enough delays the positive equilibrium is globally attractive, see, for example, [12,16,18] and references therein. The purpose of the present paper is to illustrate that the situation may change when two or more delays are involved in the same nonlinear function. This is possible when the function increases in some arguments and decreases in the others. The situation when the same production function involves different delays is quite common in real-world models. Two types of delays occur, for example, in gene regulatory systems where the translation delay and transcription delays involved in the function significantly differ. Such delays can lead to chaotic oscillations, see, for example, [10].

The presence of several delays instead of one delay can create a new type of dynamics: an equation which was stable for coinciding delays can become unstable, once the two delays are different, creating sustainable oscillations. The maximal delay value will influence the oscillation amplitude.

In [7], we presented an example of the modified Mackey–Glass equation with two delays

$$\dot{x}(t) = \frac{2x(h(t))}{1 + x^2(g(t))} - x(t), \quad t \geq 0, \tag{1.4}$$

which has the unique positive equilibrium $x = 1$, the function $f(x) = 2x/(1 + x^2)$ is increasing on $[0, 1]$, so any positive solution of (1.4) with $h \equiv g$ satisfies $\lim_{t\to\infty} x(t) = 1$, for example, by [9, Theorem 3.13] or [12, Theorem 3.3]. However, the situation changes if the two delays are different. In [7, Example 1.1] we presented an example of h, g in (1.4) such that the solution experiences sustainable oscillations.

Typically, if a nonlinear equation with one delay has two equilibrium solutions, one trivial and one positive, any nonoscillatory about the equilibria solution tends to an equilibrium (or $\pm\infty$). However, the situation changes when two delays are involved in the nonlinear production function, as Example 1.1 illustrates.

Example 1.1. The equation

$$\dot{x}(t) = \frac{3x(h(t))}{1 + \sqrt{x(g(t))}} - 2\sqrt{x}, \ t \geq 0, \quad x(t) = \varphi(t) \geq 0, \ t \leq 0 \tag{1.5}$$

has the zero equilibrium and the unique positive equilibrium $K = 4$. Let us note that the initial value problem

$$\dot{x}(t) = 12 - 2\sqrt{x}, \tag{1.6}$$

$x(0) = 9$ satisfies $x(a) = 16$, where $a \approx 1.43279$, while for the initial value problem