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Feynman–Kac theorem in random environments and partial integro-differential equations

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ABSTRACT

We prove a Feynman–Kac-type theorem for jump-diffusion models in random environments. We consider the Cauchy and Dirichlet problems. Our results enable us to calculate some conditional expectations using related partial integro-differential equations (PIDEs) and vice versa to solve some PIDEs by stochastic methods. So, the results may have many applications. We illustrate the use of our results on an example of a generalized exponential Lévy model with regime-switching.

1. Introduction

The classical Feynman–Kac theorem, proven decades ago, states that solutions of certain PDE problems are connected with functionals of Brownian motion. From that time on it has been generalized in various respects to Markovian diffusions given as solutions of stochastic differential equations (SDEs) driven by a Brownian motion (see Karatzas and Shreve [9]) and also to diffusions with jumps (see Applebaum [1]). The Feynman–Kac theorem provides a bridge between deterministic problems and stochastic ones, and it is important from both probabilistic and analytical point of view. From the probabilistic point of view the theorem says that the distributions of certain functionals of stochastic processes can be derived with the help of PDE theory. On the other hand, looking from the analytical point of view, it allows one to find solutions of certain PDE problems by means of stochastic methods. In this paper we aim at generalizing the Feynman–Kac theorem to general jump-diffusion processes in random environments. This class of processes is given by the solutions of very general SDEs (2.3) with coefficients depending on an additional "environment" process, say C. This process takes values in some finite state space and the transition intensities governing its evolution depend on the solution process. Moreover, jumps of the solution process are possible

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at transition times. This model, described in our earlier paper [7], is a generalization of diffusions with state dependent switching studied by Xi [18] and Xi and Yin [19] amongst others. Applications of these results in mathematical finance to pricing and hedging are presented in the follow-up paper [8].

This paper is organized as follows: Section 2 presents some preliminary results on the aforementioned class of processes. In Section 3 we formulate and prove a Feynman–Kac type theorem for processes that are components of a weak solution to the SDE (2.3). We connect the problem of calculating conditional expectations of some natural functionals of these processes with solving the corresponding Cauchy problem (3.3)–(3.4) or Dirichlet problem (3.25)–(3.26). Our first result (Theorem 3.2) shows that some conditional expectations of functionals of processes that are components of a weak solution of (2.3) provide solutions of the Cauchy problem. The next results, converse in a sense, appear in Theorems 3.5, 3.6, 3.7 and 3.8, and they show that, under a different set of assumptions, solutions of certain Cauchy/Dirichlet problems provide formulae for interesting conditional expectations of functionals of stochastic processes given by (2.3). In particular our Theorem 3.5 generalizes, in many respects, Theorem 3.2 in Zhu, Yin and Baran [20]. The assumptions we make in these sections are in the spirit of those in [9], where only diffusions are considered. In Section 4 we illustrate the use of our results on an example of a generalized exponential Lévy model with regime-switching, finding a classical solution to a system of PIDEs and giving the characteristic function of the log-process. In the Appendix we present Itô's lemma for general stopped semimartingales (Lemma 5.1).

2. Preliminaries on solutions of SDEs defining jump-diffusions in random environments

In this section we recall the results on SDEs in random environments (see Jakubowski and Niewęgłowski [7]) and we present two auxiliary lemmas giving semimartingale decompositions of $(v(t, Y_t, C_t))_{t\in[0,T^*]}$, where v is a nice function, and (Y, C) is a solution to (2.3) taking values in an invariant set.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. We have on this space a standard r-dimensional Wiener process W, a Poisson random measure $\pi(dx, du)$ on $\mathbf{R}^n \times [0, T]$, and counting point processes $N^{i,j}$, $i, j \in \mathcal{K}$, $i \neq j$, where $\mathcal{K}$ is a finite set, so we can take $\mathcal{K} := \{1, \ldots, K\}$. Suppose π has intensity measure $\rho(dx)du$, where ρ is a Lévy measure, so $\widetilde{\pi}(dx, du) := \pi(dx, du) - \rho(dx)du$ is a compensated Poisson random measure. Moreover, we require that the Poisson random measure π and the processes $N^{i,j}$, $i, j \in \mathcal{K}$, $i \neq j$, have no common jumps, i.e., for every $u \in [0, T]$ and every $b > 0$,

$$\int_0^u \int_{|x|>b} \Delta N_v^{i,j} \pi(dx, dv) = 0 \quad \mathbb{P}\text{-a.s.}, \tag{2.1}$$

and for all $(i_1, j_1) \neq (i_2, j_2)$,

$$\Delta N_u^{i_1,j_1} \Delta N_u^{i_2,j_2} = 0 \quad \mathbb{P}\text{-a.s.} \tag{2.2}$$

We use the convention from stochastic integration theory of writing $\int_t^u$ instead of $\int_{(t,u]}$ in stochastic and Lebesgue integrals.

Let measurable functions $\Xi : [0, T] \times \mathbf{R}^d \times \mathcal{K} \to \mathbf{R}^d$, $\Sigma : [0, T] \times \mathbf{R}^d \times \mathcal{K} \to \mathbf{R}^{d\times r}$, $\Gamma : [0, T] \times \mathbf{R}^d \times \mathcal{K} \times \mathbf{R}^n \to \mathbf{R}^d$ and $\Psi^{i,j} : [0, T] \times \mathbf{R}^d \to \mathbf{R}^d$, $i, j \in \mathcal{K}$, $i \neq j$ define SDE on $[t, T]$ of the form

$$\begin{aligned} dY_u = {} & \Xi(u, Y_u, C_u)du + \Sigma(u, Y_{u-}, C_{u-})dW_u + \int_{|x|\le a} \Gamma(u, Y_{u-}, C_{u-}, x)\widetilde{\pi}(dx, du) \\ & + \int_{|x|>a} \Gamma(u, Y_{u-}, C_{u-}, x)\pi(dx, du) + \sum_{\substack{i,j=1 \\ j\neq i}}^{K} \Psi^{i,j}(u, Y_{u-})\mathbb{1}_{\{i\}}(C_{u-})dN_u^{i,j}, \end{aligned} \tag{2.3}$$