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Equilibria with discontinuous preferences: New fixed point theorems ☆

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ABSTRACT

We prove new equilibrium existence results for games and economies with discontinuous and non-ordered preferences. To do so, we introduce the notion of "continuous inclusion property", and prove new fixed point theorems which extend and generalize the results of Fan [17], Glicksberg [20], Browder [7], and Gale and Mas-Colell [19]. Our results also extend the previous work of Yannelis [47] and Wu and Shen [46].

1. Introduction

Since the pioneer works of Dasgupta and Maskin [14] and Reny [34] on the existence of Nash equilibria in games with discontinuous payoffs, a number of authors have extended their results in different directions, see for example, Lebrun [26], Bagh and Jofre [2], Monteiro and Page [29], Bich [4,5], Bich and Laraki [6], Carbonell-Nicolau [8], Carbonell-Nicolau and McLean [9], Carmona [10–12], Prokopovych [31–33], de Castro [16], Reny [34,35,37,38], Nessah and Tian [30], Scalzo [39,41], Tian [43], Uyanık [44], and He and Yannelis [21,23,24].[1]

In this paper, we provide new equilibrium existence results for discontinuous games which are not covered by the above literature. To this end, we introduce the notion of "continuous inclusion property", which allows us to prove two new fixed point theorems. The correspondences satisfying the continuous inclusion property

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[1] See He and Yannelis [22] and Carmona and Podczeck [13] for additional references. Interested readers can consult Reny [36] for the discussions about the state of the art on the subject.

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could be neither lower nor upper hemicontinuous, actually they may be discontinuous. The continuous inclusion property is a very weak condition in the sense that any correspondence, which has either an open graph, or open lower sections, or the local intersection property[2] or it is upper hemicontinuous, will automatically satisfy this property. Our first result is an extension of fixed point theorems of Fan [17] and Glicksberg [20], which also generalizes the Browder [7] fixed point theorem in locally convex spaces. The second result generalizes substantially the fixed point theorem of Gale and Mas-Colell [19].

With the help of the above two fixed point theorems, we prove several new results. Firstly, we show the nonemptiness of demand correspondences for non-ordered and discontinuous preferences. This result generalizes the theorem of Sonnenschein [42]. Secondly, we prove the existence of Nash equilibrium for discontinuous games with non-ordered preferences. This extends the results in Reny [34] to non-ordered preferences. Thirdly, we extend the classical Gale–Debreu–Nikaido lemma (see Debreu [15]) by allowing for discontinuous demand correspondences. Our extension generalizes the Gale–Debreu–Nikaido lemma to infinite dimensional spaces, and also extends the results of Aliprantis and Brown [1] and Yannelis [47]. To show that our generalization is non-vacuous, an example of a Walrasian equilibrium with discontinuous preferences is provided, which cannot be covered by any existence result in the literature. However, our version of the Gale–Debreu–Nikaido lemma can be applied to this example.

The rest of the paper is organized as follows. In Section 2, the "continuous inclusion property" is proposed, and then we prove a fixed-point theorem and a generalization of the fixed-point theorem of Gale and Mas-Colell [19]. The existence of Nash equilibrium in games with discontinuous preferences is obtained as a direct corollary. Section 3 collects the generalization of the Gale–Debreu–Nikaido lemma to the setting with discontinuous preferences in infinite dimensional spaces.

2. Results

2.1. Definitions

Let X and Y be linear topological spaces. Suppose that ψ is a correspondence from X to Y. Then ψ is said to be **upper hemicontinuous** if the upper inverse $\psi^u(V) = \{x \in X : \psi(x) \subseteq V\}$ is open in X for every open subset V of Y, and **upper demicontinuous** if the upper inverse of every open half space in Y is open in X. The correspondence ψ is said to be **lower hemicontinuous** if the lower inverse $\psi^l(V) = \{x \in X : \psi(x) \cap V \neq \emptyset\}$ is open in X for every open subset V of Y. In addition, if $\psi^l(y) = \{x \in X : y \in \psi(x)\}$ is open for each $y \in Y$, then ψ is said to have **open lower sections**. At some $x \in X$, if there exists an open set O_x such that $x \in O_x$ and $\cap_{x' \in O_x} \psi(x') \neq \emptyset$, then we say that ψ has the local intersection property. Furthermore, ψ is said to have the **local intersection property** if this property holds for every $x \in X$.[3] Given a linear topological space X, its dual is the space X^* of all continuous linear functionals on X.

We now introduce the following "continuous inclusion property".

Definition 1. A correspondence ψ from X to Y is said to have the **continuous inclusion property** at x if there exists an open neighborhood O_x of x and a nonempty correspondence $F_x : O_x \to 2^Y$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and coF_x[4] has a closed graph.[5]

[2] See Wu and Shen [46].

[3] A continuous selection exists if a correspondence has open lower sections (see Yannelis and Prabhakar [48]) or the local intersection property (see Wu and Shen [46]) under certain convexity conditions. Scalzo [40] proposed the "local continuous selection property", which is necessary and sufficient for the existence of a continuous selection.

[4] For a correspondence F, co$F$ is the convex hull of F.

[5] If the sub-correspondence F_x has a closed graph and X is finite dimensional, then coF_x still has a closed graph since the convex hull of a closed set is closed in finite dimensional spaces. However, this may not be true if one works with infinite dimensional spaces. One can easily see that assuming the sub-correspondence F_x is convex valued and has a closed graph would suffice for our aim.