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Optimization problem for a portfolio with an illiquid asset: Lie group analysis

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A R T I C L E I N F O

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A B S T R A C T

Management of a portfolio that includes an illiquid asset is an important problem of modern mathematical finance. One of the ways to model illiquidity among others is to build an optimization problem and assume that one of the assets in a portfolio cannot be sold until a certain finite, infinite or random moment of time. This approach arises a certain amount of models that are actively studied at the moment. Working in the Merton's optimal consumption framework with continuous time we consider an optimization problem for a portfolio with an illiquid, a risky and a risk-free asset. Our goal in this paper is to carry out a complete Lie group analysis of PDEs describing value function and investment and consumption strategies for a portfolio with an illiquid asset that is sold in an exogenous random moment of time with a prescribed liquidation time distribution. The problem of such type leads to three dimensional nonlinear Hamilton–Jacobi–Bellman (HJB) equations. Such equations are not only tedious for analytical methods but are also quite challenging form a numeric point of view. To reduce the three-dimensional problem to a two-dimensional one or even to an ODE one usually uses some substitutions, yet the methods used to find such substitutions are rarely discussed by the authors.
We find the admitted Lie algebra for a broad class of liquidation time distributions in cases of HARA and log utility functions and formulate corresponding theorems for all these cases. We use found Lie algebras to obtain reductions of the studied equations. Several of similar substitutions were used in other papers before whereas others are new to our knowledge. This method gives us the possibility to provide a complete set of non-equivalent substitutions and reduced equations.

1. Introduction

Study of optimization problems with an illiquid asset leads to three dimensional nonlinear Hamilton–Jacobi–Bellman (HJB) equations. Such equations are not only tedious for analytical methods but are also

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quite challenging form a numeric point of view. One of the standard techniques is to find an inner symmetry of such equation and reduce the number of variables at least to two or if possible to one. The problems of lower dimensions are usually better studied and are, therefore, easier to handle.

All papers known to us devoted to three dimensional HJB equations provide variable substitutions without any remark on how to get similar substitution in other cases or why they use this or that substitution. Yet since the famous work of S. Lie [7] it is well known that smooth point transformations with continuous parameter admitted by linear or nonlinear partial differential equations (PDEs) can be found algorithmically using Lie group analysis. The procedure that helps to find a symmetry group admitted by a PDE is well described in many textbooks, for example, we can recommend [13,6] or [1] to the interested reader. Yet practical application of these procedures is connected with tedious and voluminous calculations which can be only slightly facilitated with the help of modern computer packages. For example, preparing this paper we used the program **IntroToSymmetry** to obtain the determining system of equations. Solving these determining systems of partial differential equations is usually rather hard and can rarely be done by an algorithm, but the possibility to find the system of determining equations facilitates the work of a researcher since the systems are quite voluminous. For example, in the studied cases the systems had more then a hundred equations.

Once the Lie algebra admitted by the studied PDE is found one can find all non-equivalent variable substitutions which reduce the dimension of the given PDE, if there are any. Using the corresponding exponential map of the adjoint representation of the admitted Lie algebra we can find the symmetry group or subgroups of the equation as well. We do not have to look for an explicit form of the admitted symmetry group to find reductions of the studied PDEs and invariant solutions of the equations. It is enough to know and to use the properties of the admitted Lie algebra which corresponds to the symmetry group. The optimal system of subalgebras of this algebra gives rise to a complete set of non-equivalent substitutions and as a result a set of different reductions of the studied PDE. The idea to develop an optimal system of subalgebras for a given symmetry algebra of a differential equation in order to find all classes of non-equivalent group invariant solutions was first introduced in [14]. This idea was used later for different PDEs and systems of ODEs arising in physics or engineering to describe the complete set of group invariant solutions. We address the reader interested in PDEs describing gas dynamics to [15], applications of this approach to engineering are discussed in [8]. In [9] the authors work with a reaction–diffusion equation with delay, and in [11] with a system of two autonomous nonlinear second-order ordinary differential equations.

The solutions of reduced PDEs are called invariant solutions because they are invariant under the action of a given subgroup. The goal of this paper is to find the admitted Lie algebras for PDEs describing value function and investment and consumption strategies for a portfolio with an illiquid asset that is sold in a random moment of time with a prescribed liquidation time distribution. We find the admitted Lie algebras for a certain class of liquidation time distributions in cases of HARA and log utility functions and formulate corresponding theorems. We provide the optimal system of subalgebras for a general case of a liquidation time distribution in both cases of HARA and logarithmic utility functions. We separately regard a case of an exponential distribution of a liquidation time where the corresponding PDE admits an extended Lie algebra. It leads to certain distinguishing properties that give rise to non-trivial reductions of three dimensional PDEs to two dimensional equations and even to ordinary differential equations in some cases. We describe all non-equivalent substitutions, provide the reductions and the corresponding lower dimensional equations as well as the corresponding allocation-consumption strategies.

2. Economical setting

In [3,2] the authors described in detail an optimization problem that corresponds to the following situation: an investor has an illiquid asset that has some paper value and can not be sold till some moment of time that is random with a prescribed distribution. He tries to maximize his average consumption investing

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