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Existence of ground state eigenvalues for the spin–boson model with critical infrared divergence and multiscale analysis

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ABSTRACT

A two-level atom coupled to the radiation field is studied. First principles in physics suggest that the coupling function, representing the interaction between the atom and the radiation field, behaves like $|k|^{-1/2}$, as the photon momentum k tends to zero. Previous results on non-existence of ground state eigenvalues suggest that in the most general case binding does not occur in the spin–boson model, i.e., the minimal energy of the atom–photon system is not an eigenvalue of the energy operator. Hasler and Herbst have shown [12], however, that under the additional hypothesis that the coupling function be off-diagonal – which is customary to assume – binding does indeed occur. In this paper an alternative proof of binding in case of off-diagonal coupling is given, i.e., it is proven that, if the coupling function is off-diagonal, the ground state energy of the spin–boson model is an eigenvalue of the Hamiltonian. We develop a multiscale method that can be applied in the situation we study, with the help of a key symmetry operator which we use to demonstrate that the most singular terms appearing in the multiscale analysis vanish.

1. Introduction

The precise description of nonrelativistic matter in interaction with the quantized radiation field has been in the focus of mathematical research ever since the proposal of the quantization of the radiation field by Dirac more than eighty years ago [9].

The invention of the Laser some fifty years ago necessitated the development of a simplified, yet, adequate model for the description of its mechanism in theoretical physics. It proved useful to simplify the model of matter from atom and molecules to two-level atoms. The corresponding model, known as the *spin–boson*

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model, became the work horse of quantum optics and is nowadays of key importance for quantum computing, with the interpretation of the two-level atom as a qubit.

Starting more than twenty years ago, the mathematical aspects of the models of nonrelativistic matter coupled to the quantized radiation field – known as *nonrelativistic quantum electrodynamics, NR QED* – were systematically investigated. In contrast to many models from relativistic quantum mechanics or quantum field theory, the models in NR QED are defined by a self-adjoint, semi-bounded Hamiltonian $H = H^* \geq c > -\infty$ acting on the tensor product $\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{F}$ of the Hilbert spaces $\mathcal{H}_{at}$ of matter and $\mathcal{F}$ of the radiation field, respectively. During the past two decades or so, for many models of NR QED, basic spectral properties like *binding* and the *existence of resonances* have been established. These represent the expected fate of the eigenvalues of the atom as it is coupled to the radiation field: The lowest spectral point persists to be an eigenvalue and all other atomic eigenvalues are unstable and give rise to metastable states, the resonances.

Specifically, *binding* means that the infimum $E_{\rm gs} := \inf \sigma(H) > -\infty$ of the spectrum of the Hamiltonian is an eigenvalue, called the ground state energy, with an eigenvector $\varphi_{\rm gs} \in \mathcal{H}$, called the ground state, i.e., $H\varphi_{\rm gs} = E_{\rm gs}\varphi_{\rm gs}$.

Binding in NR QED was established for atoms and molecules coupled to the radiation field [4,5], as well as, for the spin–boson model [1] about twenty years ago under the assumption that the coupling function $G(k)$ is slightly more regular, $|G(k)| \leq C|k|^{-\frac{1}{2}+\mu}$, for some $C < \infty$ and $\mu > 0$, in the infrared limit $k \to 0$, than what derives from first principles in physics, namely, $|G(k)| \sim C|k|^{-\frac{1}{2}}$, as $k \to 0$.

For these latter, more singular models, with $|G(k)| \sim C|k|^{-\frac{1}{2}}$, as $k \to 0$, binding was shown to hold true a few years later [7] in the special, but physically most relevant case that the radiation field is minimally coupled to the electrons of the atom. Here, it was used that the model possesses additional symmetries such as the $U(1)$-gauge symmetry. The key identity (in the case of one electron, as for the hydrogen atom) made use of in the proof is $\vec{v} = i[H, \vec{x}]$, where $\vec{v} = -i\vec{\nabla}_x - \vec{A}(\vec{x})$ is the velocity operator and $\vec{x}$ the position operator of the electron.

Following an argument of Fröhlich [11] it was assumed for many years [2] that the spin–boson model with singular coupling does not bind in the above sense, but rather possesses a ground state that is revealed by a (non-unitary) change of the representation of the canonical commutation relations. In view of this common belief the recent proof of Hasler and Herbst [12] for binding of the spin–boson model with singular coupling is a remarkable result. Their proof uses the renormalization group based on the isospectral Feshbach–Schur map developed in [5,6,3]. Their additional key observation is that since there is no self-interaction of each of the two levels of the atom, but only a coupling to one another, the (discrete) flow equation defined by the renormalization group is more regular than it seems to be at first glance.

In the present paper we give an alternative proof for binding of the spin–boson model with singular coupling. We consider the spin–boson Hamiltonian

$$H \ := \ H_{at} + H_{\rm ph} + \Phi(G), \tag{1.1}$$

where $H_{\rm ph} \equiv \mathbf{1}_{at} \otimes H_{\rm ph}$ is the field Hamiltonian and $H_{at} = \sigma_3 + \mathbf{1}_{at} \equiv H_{at} \otimes \mathbf{1}_{\mathcal{F}}$ is the Hamiltonian of the two-level atom, with σ_ν denoting the Pauli matrices. Furthermore, $\Phi(G)$ is the interaction with field operator $\Phi(G) = a^*(G) + a(G)$, with $G \equiv g\sigma_1 \otimes h(k)$, where h is a compactly supported coupling function obeying $|h(k)| \sim c|k|^{-\frac{1}{2}}$, as $k \to 0$, and $g \geq 0$ is the coupling strength, see Eqs. (1.16)–(1.18). For this Hamiltonian H we prove our main result, Theorem 1.1, which states that the infimum of its spectrum is an eigenvalue.

Our construction is based on *Pizzo's method* [16], rather than the renormalization group induced by the Feshbach–Schur map. That is, we consider a sequence $H_n \equiv H(G_n)$ of regularized Hamiltonians whose coupling functions $G_n(k) = \mathbf{1}(|k| \geq \rho_n)G(k)$ are the restrictions of G to photon momenta larger than $\rho_n = \kappa\gamma^n$, for some fixed $\gamma < 1$ and all $n \in \mathbb{N}$. Following the idea originally formulated by Pizzo, we