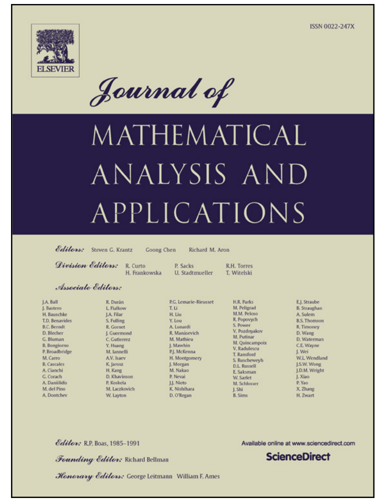

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LOCALLY UNIVALENT APPROXIMATIONS OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we introduce a measure of the non-univalency of an analytic function, and we use it in order to find the best approximation of analytic function by a locally univalent normalized analytic function.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad z \in \Delta, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ will be denoted by $\mathcal{U}$.

Following [3], for $\alpha \in \mathbb{R}$ we consider the class $\mathcal{G}(\alpha)$ consisting of locally univalent functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}, \qquad z \in \Delta. \tag{1.2}$$

It is easy to see that the identity function satisfies the above inequality for $\alpha > 0$, thus $\mathcal{G}(\alpha) \neq \varnothing$ if $\alpha > 0$, and we will make this assumption on α in the sequel. In [4], Ozaki introduced the class $\mathcal{G} \equiv \mathcal{G}(1)$ and proved that functions in $\mathcal{G}$ are univalent in Δ. In [13], Umezawa generalized Ozaki's result for a version of the class $\mathcal{G}$ (convex functions in one direction). It is also known that the functions in the class $\mathcal{G}(1)$ are starlike in Δ (see for example the particular case $\alpha = 1$ of (16) in [9], or [11], [12]).

Since $\mathcal{G}(\alpha) \subset \mathcal{G}(\alpha')$ whenever $\alpha < \alpha'$, it readily follows that the class $\mathcal{G}(\alpha)$ is included in the class $\mathcal{S}$ of starlike functions in Δ whenever $\alpha \in (0, 1]$, which in particular shows that $\mathcal{G}(\alpha)$ consists only of univalent functions for any $\alpha \in (0, 1]$. In the present paper we will investigate the properties of the class $\mathcal{G}(\alpha)$ (and of a certain subclass $\mathcal{G}^*(\alpha)$ of it), and then we will determine the best approximation of an analytic function by functions in the class $\mathcal{G}(\alpha)$ in the sense of L^2 norm. The method is based on solving a certain semi-infinite quadratic problem, in the spirit of [7] and [8].

The structure of the paper is the following. In Section 2 we introduce the subclass $\mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$, defined by a certain inequality in terms of the Taylor coefficients of the function. Next, we investigate the connection between the class $\mathcal{G}(\alpha)$ (for various values of $\alpha > 0$) and the classical classes of starlike and convex functions (Theorem 2.1).

As indicated above, it is an open problem whether $\mathcal{G}(\alpha) \subset \mathcal{U}$ for $\alpha > 1$. In Theorem 2.2 we give a partial result for this problem, which shows that for $\alpha \in [1, 4.952)$ the radius of univalence of the class $\mathcal{G}(\alpha)$ is at least $1/\alpha$. The section concludes with a result (Proposition 2.1) which shows that for certain values of $\alpha \in (0, 1]$ the class $\mathcal{G}^*(\alpha)$ interpolates between subclasses of starlike and convex functions, and that the result is sharp.

In order to investigate the problem of the best approximation of an analytic function by functions in the class $\mathcal{G}^*(\alpha)$ (in the sense of L^2 norm), in Section 3 we introduce and solve a semi-infinite quadratic programming problem, which may be of independent interest (Theorem 3.2). The paper concludes with Section 4, in which we apply the results of the previous section in order to settle the problem of the best approximation of an analytic function by functions in the class $\mathcal{G}^*(\alpha)$ (Theorem 4.1), and to present some numerical examples (Example 4.1).

2. Results on the classes $\mathcal{G}(\alpha)$ and $\mathcal{G}^*(\alpha)$

It can be easily seen that functions in $\mathcal{G}(\alpha)$ are not necessarily univalent in Δ if $\alpha > 1$, as shown by the following example.

Example 2.1. *Consider the function $f : \Delta \to \mathbb{C}$ defined by $f(z) = \frac{1}{3}(z-1)^3 + \frac{1}{3}$, $z \in \Delta$. It is easy to see that the function f belongs to the class $\mathcal{A}$ and it is locally univalent. Since*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(1 + \frac{2z}{z-1}\right) < 2 = 1 + \frac{2}{2}, \qquad z \in \Delta,$$

it follows that $f \in \mathcal{G}(2)$.

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