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Kazdan–Warner equation on graph in the negative case

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ABSTRACT

Let $G = (V, E)$ be a connected finite graph. In this short paper, we reinvestigate the Kazdan–Warner equation

$$\Delta u = c - he^u$$

with $c < 0$ on G, where h defined on V is a known function. Grigor'yan, Lin and Yang [3] showed that if the Kazdan–Warner equation has a solution, then $\overline{h}$, the average value of h, is negative. Conversely, if $\overline{h} < 0$, then there exists a number $c_-(h) < 0$, such that the Kazdan–Warner equation is solvable for every $0 > c > c_-(h)$ and it is not solvable for $c < c_-(h)$. Moreover, if $h \leq 0$ and $h \not\equiv 0$, then $c_-(h) = -\infty$. Inspired by Chen and Li's work [1], we ask naturally:

Is the Kazdan–Warner equation solvable for $c = c_-(h)$?

In this paper, we answer the question affirmatively. We show that if $c_-(h) = -\infty$, then $h \leq 0$ and $h \not\equiv 0$. Moreover, if $c_-(h) > -\infty$, then there exists at least one solution to the Kazdan–Warner equation with $c = c_-(h)$.

1. Introduction

It is well known that the following equation

$$\Delta_g u = c - he^u$$

gives a description of the conformal deformation of the smooth metric g on a 2-dimensional closed Riemannian manifold (M, g). Kazdan and Warner had given satisfying characterizations to the solvability of the above equation.

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Grigor'yan, Lin and Yang [3] first studied the corresponding equation on a connected finite graph $G = (V, E)$, where V is the vertex set and E is the edge set. Denote $C(V)$ as the set of real functions on V. The discrete graph Laplacian $\Delta : C(V) \to C(V)$ is

$$\Delta f_i = \frac{1}{\mu_i}\sum_{j\sim i}\omega_{ij}(f_j - f_i)$$

for $f \in C(V)$ and $i \in V$, where $\mu : V \to (0, +\infty)$ is a fixed vertex measure, and $\omega : E \to (0, +\infty)$ is a fixed symmetric edge measure on G. Grigor'yan, Lin and Yang [3] considered

$$\Delta u = c - he^u, \tag{1.1}$$

where $c \in \mathbb{R}$, and $h \in C(V)$. For any $f \in C(V)$, denote $\overline{f}$ as the average value of f with respect to the measure μ. Let us summarize the results of Grigor'yan, Lin and Yang [3].

- $c = 0$ case. Assume $h \not\equiv 0$, then the equation (1.1) has a solution if and only if h changes sign and $\overline{h} < 0$.
- $c > 0$ case. The equation (1.1) has a solution if and only if h is positive somewhere.
- $c < 0$ case. If (1.1) has a solution, then $\overline{h} < 0$. On the contrary, if $\overline{h} < 0$, then there exists a constant $-\infty \le c_-(h) < 0$ depending only on h such that (1.1) has a solution for any $c \in (c_-(h), 0)$, but has no solution for any $c < c_-(h)$. Moreover, if $h \le 0$ and $h \not\equiv 0$, then $c_-(h) = -\infty$ and hence (1.1) always has a solution.

In the following we shall call the equation (1.1) "Kazdan–Warner equation" (on graph). One can see from Grigor'yan, Lin and Yang's results, when c is nonnegative, the solvability of (1.1) has been understood completely. However, in the case when c is negative, one still needs to know:

Can one solve the Kazdan–Warner's equation (1.1) when $c = c_-(h)$?

The main purpose of this paper is to answer the above question. We prove

Theorem 1.1. *Consider the Kazdan–Warner equation (1.1) with $c < 0$ and $\overline{h} < 0$. Suppose that $c_-(h) < 0$ is given as above. If $c_-(h) = -\infty$, then $h \le 0$ and $h \not\equiv 0$. If $c_-(h) > -\infty$, then there exists at least one solution to (1.1) with $c = c_-(h)$.*

In the following of this paper, we prove Theorem 1.1 by using variational principles and the method of upper and lower solutions. We follow the approach pioneered by Chen and Li [1], and Kazdan and Warner [4].

2. The proof of Theorem 1.1

Lemma 2.1. *Consider the Kazdan–Warner equation (1.1) with $c < 0$. If it has a solution u, then the unique solution φ to*

$$(\Delta + c)\varphi = h \tag{2.1}$$

satisfies $\varphi \ge e^{-u}$.

Proof. Using $e^x - 1 \ge x$ for any $x \in \mathbb{R}$, we have $e^{u_i - u_j} - 1 \ge u_i - u_j$ and further

$$\omega_{ij}(e^{-u_j} - e^{-u_i}) \ge e^{-u_i}\omega_{ij}(u_i - u_j).$$