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Currents carried by the graphs of semi-monotone maps

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ABSTRACT

In this paper we study the structure, weak continuity and approximability properties for the integer multiplicity rectifiable currents carried by the graphs of maximal semi-monotone set-valued maps on an n-dimensional convex domain. Especially, we give an enhanced version of approximation theorem for the subgradients of semi-convex functions.

1. Introduction and main results

This paper is devoted to a study of some properties of the graphs of maximal semi-monotone set-valued maps, defined on a bounded open convex set $\Omega \subset \mathbb{R}^n$, in terms of Cartesian current.

Giaquinta–Modica–Souček [7] introduced a class of functions $u \in L^1(\Omega, \mathbb{R}^n)$, named $\mathcal{A}^1(\Omega, \mathbb{R}^n)$, such that u is approximately differentiable a.e. and all minors of the Jacobian matrix Du are summable in Ω. For $u \in \mathcal{A}^1(\Omega, \mathbb{R}^n)$, it is well defined an integer multiplicity (i.m.) rectifiable current G_u carried by the rectifiable graph of u. More precisely,

$$G_u = \tau(\mathcal{G}_{u,\Omega}, 1, \xi_u).$$

Here the rectifiable graph of u is defined by

$$\mathcal{G}_{u,\Omega} = \{(x, u(x)) \mid x \in \mathcal{L}_u \cap A_D(u) \cap \Omega\},$$

where $\mathcal{L}_u$ is the set of Lebesgue points, $u(x)$ is the Lebesgue value, and $A_D(u)$ is the set of approximate differentiability points of u. The unit n-vector ξ_u given at each point $(x, u(x)) \in \mathcal{G}_{u,\Omega}$ provides an orientation

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to the approximate tangent space $Tan^n(\mathcal{G}_{u,\Omega}, x)$, for more details see [5,7]. Moreover, the area of $\mathcal{G}_{u,\Omega}$, denoted by $A(u, \Omega)$, is equal to the mass of G_u, i.e.,

$$A(u, \Omega) = \mathbf{M}(G_u).$$

The initial motivation of our work is the following: Alberti–Ambrosio [1] defined an n-current in $\mathbb{R}^n \times \mathbb{R}^n$ for maximal monotone set-valued maps on $\mathbb{R}^n$, and studied weak continuity and approximation properties for such currents. A natural problem is raised whether we can extend the definitions and results for $\mathcal{A}^1(\Omega, \mathbb{R}^n)$ and maximal monotone maps to maximal semi-monotone maps, an important class of set-valued maps which include the subgradients of semi-convex functions. For this problem, we [13] gave some positive answers. More precisely, we defined an i.m. rectifiable current $G_F := \tau(\Gamma_{F,\Omega}, 1, \xi)$ carried by the graph (denoted by $\Gamma_{F,\Omega}$) of the maximal semi-monotone map F on Ω (i.e. $F \in MSM(\Omega)$, see Definition 2.1) such that the current has zero boundary (i.e. $\partial G_F \llcorner \Omega \times \mathbb{R}^n = 0$) and the orientation ξ in "nonvertical parts" is consistent with the one given in the class $\mathcal{A}^1(\Omega, \mathbb{R}^n)$. Here we will study the structure properties for the current G_F. We refer to Sec. 2 below for the notation.

Theorem 1.1. *Let $F \in MSM(\Omega)$ and a single-valued map $f : \Omega \to \mathbb{R}^n$ such that $f(x) \in F(x)$ for any $x \in \Omega$. Then*

(i) $G_F \llcorner \Omega' \times \mathbb{R}^n \in \operatorname{cart}(\Omega' \times \mathbb{R}^n)$ *for any open subset* $\Omega' \subset\subset \Omega$.
(ii) $f \in \mathcal{A}^1_{loc}(\Omega, \mathbb{R}^n) \cap BV_{loc}(\Omega, \mathbb{R}^n) \cap L^\infty_{loc}(\Omega, \mathbb{R}^n)$.
(iii) $G_F = G_f + S$ *where* $S := G_F \llcorner \mathcal{M}_0$, $\mathcal{M}_0 := \{(x, y) \in \Gamma_{F,\Omega} \mid \xi^{\bar{0}0}(x, y) = 0\}$.

Here the class of Cartesian currents $\operatorname{cart}(\Omega \times \mathbb{R}^n)$ (see [7, Vol. I, Sec. 4.2.2]) is defined by

$$\begin{aligned}\operatorname{cart}(\Omega \times \mathbb{R}^n) := \{T \in \mathcal{D}_n(\Omega \times \mathbb{R}^n) \mid\ & T \text{ is an i.m. rectifiable current in } \Omega \times \mathbb{R}^n \\ & \mathbf{M}(T) < \infty, \|T\|_1 < \infty, \partial T \llcorner \Omega \times \mathbb{R}^n = 0, T^{\bar{0}0} \geqslant 0, \pi_\sharp T = [\![\Omega]\!]\}.\end{aligned}$$

For $\mathcal{H}^n$-a.e. $(x, y) \in \Gamma_{F,\Omega}$, the orientation $\xi(x, y)$ can be written as (see [7, Vol. I, Sec. 2.2.1, 2.2.4])

$$\xi(x, y) = \sum_{|\alpha|+|\beta|=n} \xi^{\alpha\beta}(x, y) e_\alpha \wedge \epsilon_\beta.$$

An important problem is to characterize the Cartesian currents $T \in \operatorname{cart}(\Omega \times \mathbb{R}^n)$ for which there is a sequence of smooth maps $u_k : \Omega \to \mathbb{R}^n$ such that

$$G_{u_k} \rightharpoonup T, \qquad \mathbf{M}(G_{u_k}) \to \mathbf{M}(T),$$

where the notation $\mathbf{M}(...)$ refer to Sec. 2. Such a question is connected with the problem of relaxation of the area integral for nonparametric graphs which is discussed by Giaquinta–Modica–Souček [6] and [7, Vol. II, Sec. 6]. Then Mucci [9–12] made efforts to investigate such problem and gave some partial answers.

These facts and results lead us to introduce the notion of relaxed area of the graph of a map $F \in MSM(\Omega)$ with respect to weak convergence as currents

$$\widetilde{A}(F, \Omega') := \inf\{\liminf_{k\to\infty} A(f_k, \Omega') \mid f_k \in C^1(\Omega', \mathbb{R}^n), \quad G_{f_k} \rightharpoonup G_F \text{ in } \mathcal{D}_n(\Omega' \times \mathbb{R}^n)\},$$

where Ω' is a proper open subset of Ω and the notation $\mathcal{D}_n(...)$ refer to Sec. 2. Moreover, if we denote by $A(F, \Omega')$ the area of the graph of F on $\Omega' \times \mathbb{R}^n$, then by lower semicontinuity of mass one has