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The jump set under geometric regularisation. Part 2: Higher-order approaches

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A B S T R A C T

In Part 1, we developed a new technique based on Lipschitz pushforwards for proving the jump set containment property $\mathcal{H}^{m-1}(J_u \setminus J_f) = 0$ of solutions u to total variation denoising. We demonstrated that the technique also applies to Huber-regularised TV. Now, in this Part 2, we extend the technique to higher-order regularisers. We are not quite able to prove the property for total generalised variation (TGV) based on the symmetrised gradient for the second-order term. We show that the property holds under three conditions: First, the solution u is locally bounded. Second, the second-order variable is of locally bounded variation, $w \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, instead of just bounded deformation, $w \in \mathrm{BD}(\Omega)$. Third, w does not jump on J_u parallel to it. The second condition can be achieved for non-symmetric TGV. Both the second and third condition can be achieved if we change the Radon (or L^1) norm of the symmetrised gradient Ew into an L^p norm, $p > 1$, in which case Korn's inequality holds. On the positive side, we verify the jump set containment property for second-order infimal convolution TV (ICTV) in dimension $m = 2$. We also study the limiting behaviour of the singular part of Du, as the second parameter of TGV^2 goes to zero. Unsurprisingly, it vanishes, but in numerical discretisations the situation looks quite different. Finally, our work additionally includes a result on TGV-strict approximation in $\mathrm{BV}(\Omega)$.

1. Introduction

We introduced in Part 1 [38] the *double-Lipschitz comparability condition* of a regularisation functional R. Roughly

$$R(\overline{\gamma}_{\#}u) + R(\underline{\gamma}_{\#}u) - 2R(u) \le T_{\overline{\gamma},\underline{\gamma}}|Du|(\operatorname{cl} U), \tag{1}$$

whenever $\overline{\gamma}, \underline{\gamma} : \Omega \to \Omega$ are bi-Lipschitz transformations reducing to the identity outside $U \subset \Omega$. Constructing specific Lipschitz shift transformations around a point $x \in J_u$, for which the constant $T_{\overline{\gamma},\underline{\gamma}} = O(\rho^2)$ for $\rho > 0$ the size of the shift, we were able to prove the jump set containment

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$$\mathcal{H}^{m-1}(J_u \setminus J_f) = 0 \tag{J}$$

for $u \in \mathrm{BV}(\Omega)$ the solution of the denoising or regularisation problem

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \phi(f(x) - u(x))\, dx + R(u). \tag{P}$$

The admissible fidelities ϕ include here $\phi(t) = t^p$ for $1 < p < \infty$. For $p = 1$ we produced somewhat weaker results comparable to those for total variation (TV) in [23]. The admissible regularisers R included, obviously, total variation, for which the property was already proved previously by level set techniques [14]. We also showed the property for Huber-regularised total variation as a new contribution besides the technique. If non-convex total variation models and the Perona–Malik anisotropic diffusion were well-posed, we demonstrated that the technique would also apply to them.

The development of the new technique was motivated by higher-order regularisers, in particular by total generalised variation (TGV, [9]), for which the level set technique is not available due to the lack of a co-area formula. In this Part 2, we now aim to extend our Lipschitz pushforward technique to variants of TGV as well as infimal convolution TV (ICTV, [15]). In order to do this, we need to modify the double-Lipschitz comparability criterion (1) a little bit. Namely, we will in Section 3 introduce rigorously a *partial double-Lipschitz comparability condition* of the form

$$R(\overline{\gamma}_\#(u-v)+v) + R(\underline{\gamma}_\#(u-v)+v) - 2R(u) \le T_{\overline{\gamma},\underline{\gamma}}|D(u-v)|(\mathrm{cl}\, U) + \text{small terms}. \tag{2}$$

Here, in comparison to (1), we have subtracted v from u before the pushforward. The idea is the same as in the application the jump set containment result for TV to prove it for ICTV. As we may recall

$$\mathrm{ICTV}_{\vec{\alpha}}(u) := \min_{\substack{v\in W^{1,1}(\Omega),\\ \nabla v\in \mathrm{BV}(\Omega;\mathbb{R}^m)}} \alpha\|Du - \nabla v\|_{2,\mathcal{M}(\Omega;\mathbb{R}^m)} + \beta\|D\nabla v\|_{F,\mathcal{M}(\Omega;\mathbb{R}^{m\times m})},$$

where $\vec{\alpha} = (\beta, \alpha)$. Now, if u solves (P) for $R = \mathrm{ICTV}_{\vec{\alpha}}$, then u solves

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \phi(|f(x) - u(x)|)\, dx + \alpha\|Du - \nabla v\|_{2,\mathcal{M}(\Omega;\mathbb{R}^m)},$$

with v fixed. Otherwise written, $\bar{u} = u - v$ solves for $\bar{f} = f - v$ the total variation denoising problem

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \phi(|\bar{f}(x) - \bar{u}(x)|)\, dx + \alpha\|D\bar{u}\|_{2,\mathcal{M}(\Omega;\mathbb{R}^m)}.$$

Since $v \in W^{1,1}(\Omega)$ has no jumps, $J_{\bar{f}} = J_f$, the property (J) that ICTV would introduce no jumps would follow from the corresponding result for TV if we had further $v \in L^\infty(\Omega)$. We verify that this is indeed the case if $m \in \{1, 2\}$, and consequently prove the jump set containment property for ICTV in these dimensions.

The idea with v in (2) is roughly the same as this: to remove the second-order information from the problem, and reduce it into a first-order one. However, unlike in the case of ICTV, generally, we cannot reduce the problem to TV. Indeed, written in the differentiation cascade formulation [11], second-order TGV reads as

$$\mathrm{TGV}^2_{\vec{\alpha}}(u) := \min_{w\in \mathrm{BD}(\Omega)} \alpha\|Du - w\|_{2,\mathcal{M}(\Omega;\mathbb{R}^m)} + \beta\|Ew\|_{F,\mathcal{M}(\Omega;\mathrm{Sym}^2(\mathbb{R}^m))}. \tag{3}$$