J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Extensions of subcopulas

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A R T I C L E I N F O

Article history:
Received 12 November 2016
Available online xxxx
Submitted by U. Stadtmueller

Keywords:
Copula
Hausdorff metric
Non-continuous random vectors
Sklar's Theorem
Subcopula

A B S T R A C T

In view of Sklar's Theorem the probability distribution function of every (not necessarily continuous) random vector can be uniquely decomposed in terms of the marginal distributions of its components and a suitable subcopula. The study of such latter functions is therefore of interest for understanding the dependence information of non-continuous variables. Here, we investigate some analytical properties of the class of subcopulas, including compactness (with respect to a novel metric), approximations and Baire category results. Moreover, under a suitable assumption, we describe all possible extensions from a subcopula to a copula in any dimension.

1. Introduction

Copulas have been largely employed for describing the association of continuous random variables, as can be witnessed by several contributions devoted to the topic; see, for instance, [11,18,21,25]. Another related aspect of interest is, nowadays, the use of copulas in the determination of statistical models for a random vector $\mathbf{X} = (X_1, \ldots, X_p)$ whose components are possibly not-continuous. In this latter case, however, the copula associated with $\mathbf{X}$ is not anymore unique, a fact that needs special care in several practical problems (see, for instance, [16,22]). In fact, as can be inferred by the proof of Sklar's Theorem [31] (see also [9,12,30]), a copula associated with a non-continuous random vector $\mathbf{X}$ is uniquely determined only on the Cartesian product of the (closure of the) ranges of $X_1, \ldots, X_p$, but various extensions to the whole domain $[0, 1]^p$ are possible. In [31], the term *subcopula* was introduced to denote the function (defined on a suitable subset of $[0, 1]^p$) that contains the information about the dependence of a not-necessarily continuous random vector.

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http://dx.doi.org/10.1016/j.jmaa.2017.02.061
0022-247X/

Various methods to extend a subcopula to a copula have been considered, for instance, in [6,31], while maximal and minimal extensions can be found in [4,16]. Such methods are relevant, for instance, in defining various measures of association for non-continuous data [24,26,33]. Moreover, they can provide tools for nonparametric estimation of a copula, where it is of interest to smooth the empirical copula while preserving copula properties [13,17,27,32], and/or extend copulas defined on a discrete setting [20,23,29]. Moreover, extensions of sub-copulas may be helpful to understand the limit of copula-based inferential procedures when they are applied, without some due changes, to non-continuous data [3,19,28].

Here, we continue the study of subcopulas and their extensions by providing a general framework to deal with such problems. Specifically, we introduce a distance ξ in the class of subcopulas that is based on the Hausdorff distance of the respective graphs. As a relevant aspect, we show that the class of subcopulas equipped with the topology induced by ξ is compact (Section 2). Hence, we use continuity arguments to prove, in an alternative way, that any subcopula can be extended to a copula (Section 2.1). Finally, in Section 3 we provide the general analytical expression for all the extensions of a subcopula in a multivariate setting, which generalizes the results presented in [6] for the bivariate case.

2. A metric for subcopulas

For basic definitions and properties of copulas we refer to [11,25]. Here, we only recall the minimum bare that is necessary to make this manuscript self-contained.

Definition 2.1. Let $A_1, \dots, A_p$ be subsets of $[0,1]$ containing both 0 and 1. Then a *subcopula* is a function $S : A_1 \times \cdots \times A_p \to [0,1]$ such that

(a) $S(u_1, \dots, u_p) = 0$ if $u_j = 0$ for at least one index $j \in \{1, \dots, p\}$;
(b) $S(1, \dots, 1, t, 1, \dots, 1) = t$ for every $t \in A_j$ $(j \in \{1, \dots, p\})$;
(c) For every rectangle $[\mathbf{a}, \mathbf{b}]$ having its vertices in $A_1 \times \cdots \times A_p$, the S-volume of $[\mathbf{a}, \mathbf{b}]$ is non-negative, namely $V_S([\mathbf{a}, \mathbf{b}]) \geq 0$, where

$$V_S([\mathbf{a}, \mathbf{b}]) = \sum_{\mathbf{v} \in \mathrm{ver}[\mathbf{a}, \mathbf{b}]} \mathrm{sign}(\mathbf{v})\, S(\mathbf{v}),$$

with

$$\mathrm{sign}(\mathbf{v}) = \begin{cases} 1, & \text{if } v_j = a_j \text{ for an even number of indices,} \\ -1, & \text{if } v_j = a_j \text{ for an odd number of indices,} \end{cases}$$

and $\mathrm{ver}([\mathbf{a}, \mathbf{b}]) = \{a_1, b_1\} \times \cdots \times \{a_p, b_p\}$ is the set of vertices of $[\mathbf{a}, \mathbf{b}]$.

The class of subcopulas will be denoted by $\mathscr{S}$. Every $S \in \mathscr{S}$ is Lipschitz continuous with constant 1 (shortly 1-Lipschitz), i.e.

$$|S(\mathbf{u}) - S(\mathbf{v})| \leq \sum_{i=1}^{p} |u_i - v_i| \tag{1}$$

for every $\mathbf{u}, \mathbf{v} \in \mathrm{Dom}(S)$. Thus, the domain of S can be extended without loss of generality to its closure. In the following, if not otherwise stated, we will therefore assume that $A_1, \dots, A_p$ are closed.

A *copula* is a subcopula defined on $[0,1]^p$, namely such that $A_j = [0,1]$ for every $j \in \{1, \dots, p\}$. The class of copulas will be denoted by $\mathscr{C}$. In particular, given $S \in \mathscr{S}$, a copula C is said to be an *extension*

Please cite this article in press as: E. de Amo et al., Extensions of subcopulas, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.061