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Characterizations of ordered operator spaces

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A B S T R A C T

We demonstrate new abstract characterizations for unital and non-unital operator spaces. We characterize unital operator spaces in terms of the cone of accretive operators (operators whose real part is positive). We show that matrix norms and accretive cones are induced by gauges, although inducing gauges are not unique in general. Finally, we show that completely positive completely contractive linear maps on non-unital operator spaces extend to any containing operator system if and only if the operator space is induced by a unique gauge.

1. Introduction

The study of operator spaces and operator systems has played an increasingly important role in operator theory and operator algebras since the introduction of completely positive maps by Stinespring in [21] and the seminal work on completely positive maps by Arveson in [1]. **Operator spaces** (vector subspaces of $B(H)$, the bounded linear operators on a Hilbert space H) have been studied in the context of Ruan's Theorem (Theorem 1.4 below), which characterizes them up to complete isometry. **Operator systems** (unital self-adjoint operator spaces), on the other hand, have been studied in the context of the Choi–Effros Theorem (Theorem 1.2 below) which characterizes them up to unital complete order isomorphism. Since the norm structure of an operator system is determined by its order structure, operator systems are examples of ordered operator spaces – operator spaces possessing a specified cone of positive operators at each matrix level. Abstract operator spaces, on the other hand, lack natural cones of positive operators. For example, if $V \subset B(H)$ is a concrete operator space, then the mapping

$$x \mapsto \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

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into $B(H^2)$ is easily seen to be completely isometric. As abstract operator spaces, V and its image under the above mapping are identical, even though V could contain non-zero positive operators while its image in $B(H^2)$ does not. In other words, abstract operator spaces forget their order structure.

In this paper, we will demonstrate an abstract characterization for operator spaces in terms of matrix gauges (Theorem 4.8). Concretely, we define the gauge of an operator $T \in B(H)$ to be $\nu(T) = \|Re(T)_+\|$, where $Re(T)_+$ is the positive part of the real part of T. From the gauge, we can recover the norm (see Lemma 4.7), involution and order structure at every matrix level. Any mapping which is "completely gauge isometric" will automatically be completely isometric, self-adjoint, and a complete order embedding. In the course of proving our main result, we will also recover an abstract characterization for unital operator spaces in terms of cones of **accretive operators**, operators whose real part is positive (Theorem 3.4). As applications, we will consider representations of normal operator spaces and extensions of completely positive completely contractive maps. A normal operator space is an abstract operator space together with an order structure which satisfies the condition that $x \leq y \leq z$ implies $\|y\| \leq \max(\|x\|, \|z\|)$ at every matrix level. We provide representation theorems for these objects (Theorems 5.4 and 5.6). This is achieved by proving that the norm and order structures on a normal operator space are induced by a matrix gauge. We show by example that this inducing matrix gauge is not always unique. In fact, we can characterize operator spaces with unique inducing gauges as operator spaces with the "real-cpcc extension property". This is the property that every completely positive completely contractive linear map from the given operator space into $B(H)$ can be extended to a completely positive completely contractive map on any containing operator system. For example, operator systems have the real-cpcc extension property, by the Arveson extension theorem.

Before moving on, we briefly review some related literature. The results in sections 2 and 3 rely heavily upon the theory of accretive operators and real-completely positive maps. Cones of accretive operators have been studied in the context of operator algebras, unital operator spaces, and Banach algebras in [2,3,7–10]. See section 3 of [9] for several fundamental properties of the cone of accretive operators. The real-completely positive maps defined in section 2 were also studied by Blecher, Read and other authors. See section 2 of [2] for several fundamental results concerning these maps. The study of accretive operators and real-completely positive maps was brought to the author's attention by David Blecher. Abstract characterizations of unital operator spaces up to complete isometry can be found in [5] and [6]. Another characterization of unital operator spaces (in terms of the existence of sufficiently many unital functionals) can be found in [13]. Cones of accretive operators are not addressed in [5,6] or [13]. Matrix gauges, considered in section 3, were introduced by Effros and Winkler in [12] as non-commutative generalizations of Minkowski gauges. Effros and Winkler were able to prove analogues of the classical bipolar and Hahn–Banach Theorems for matrix gauges. We prove a special case of their Hahn–Banach Theorem using our results (Theorem 6.4). Questions about abstract operator spaces with a matricial order structure go back to the work of Schreiner who studied "matrix regular operator spaces" in [20]. The matrix regular condition is similar to, but more restrictive than, our notion of normality. For example, the positive cone of a matrix regular operator space spans the entire space, while this may not be the case in a normal operator space. Abstract characterizations of non-unital self-adjoint ordered operator spaces can also be found in [15,16,22]. Each of these authors take as an axiom the existence of sufficiently many positive functionals to norm the space, whereas we make no such assumption. Normality is also mentioned in Werner's paper [22] in connection with the existence of sufficiently many positive functionals to norm the space. An abstract characterization for matrix-ordered $*$-algebras due to Juschenko and Popovych can by found in [14].

We now summarize some basic definitions, notation, and background. For a detailed introduction to these topics, we refer the reader to [17]. We will call a vector space V a $\mathbb{R}$-vector space (respectively $\mathbb{C}$-vector space) if the underlying field is $\mathbb{R}$ (respectively, $\mathbb{C}$). For any subset S of a $\mathbb{C}$-vector space, we let "span S" denote the set of $\mathbb{C}$-linear combinations of elements of S. Let V be a ($\mathbb{R}$ or $\mathbb{C}$)-vector space. For each $n \in \mathbb{N}$, we let $M_n(V)$ denote the vector space of $n \times n$ matrices with entries in V. For each $n, m \in \mathbb{N}$, we let M_n (respectively, $M_{n,m}$) denote the $n \times n$ (respectively, $n \times m$) matrices with entries in $\mathbb{C}$. We use the notation