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Study of the period function of a two-parameter family of centers ☆

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ABSTRACT

In this paper we study the period function of $\ddot{x} = (1+x)^p - (1+x)^q$, with $p, q \in \mathbb{R}$ and $p > q$. We prove three independent results. The first one establishes some regions in the parameter space where the corresponding center has a monotonous period function. This result extends the previous ones by Miyamoto and Yagasaki for the case $q = 1$. The second one deals with the bifurcation of critical periodic orbits from the center. The third one is addressed to the critical periodic orbits that bifurcate from the period annulus of each one of the three isochronous centers in the family when perturbed by means of a one-parameter deformation. These three results, together with the ones that we obtained previously on the issue, lead us to propose a conjectural bifurcation diagram for the global behaviour of the period function of the family.

1. Introduction and setting of the problem

In the present paper we study the bifurcation diagram of the period function associated to a family of potential centers. Recall that a singular point p of an analytic vector field $X = f(x,y)\partial_x + g(x,y)\partial_y$ is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest neighbourhood with this property is called *period annulus* and henceforth it will be denoted by $\mathscr{P}$. From now on $\partial\mathscr{P}$ will denote the boundary of $\mathscr{P}$ after embedding it into $\mathbb{RP}^2$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi := \partial\mathscr{P} \setminus \{p\}$. Note that Π is a non-empty compact subset of $\mathbb{RP}^2$. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set

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of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parametrize this set. This can be done by taking an analytic transverse section to X on $\mathscr{P}$, for instance an orbit of the orthogonal vector field $X^{\perp}$. If $\{\gamma_s\}_{s\in(a,b)}$ is such a parametrization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is an analytic map that provides the qualitative properties of the period function that we are concerned about. In particular the existence of *critical periods*, which are isolated critical points of this function, i.e. $\hat{s} \in (a,b)$ such that $T'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can readily see that this definition does not depend on the particular parametrization of the set of periodic orbits used. We say that the period function of a center is *monotonous increasing* (respectively, *decreasing*) if there are no critical periodic orbits on $\mathscr{P}$ and, for any two periodic orbits γ_1 and γ_2 with $\gamma_1 \subset \mathrm{Int}(\gamma_2)$, the period of γ_2 is greater (respectively, smaller) than the one of γ_1.

The problem of bounding the number of critical periodic orbits is analogous to the problem of bounding the number of limit cycles, which is related to the well known Hilbert's 16th Problem (see [1,7,20,25] and references therein) and its various weakened versions. Questions related to the behaviour of the period function have been extensively studied by a number of authors. Let us quote for instance the problems of isochronicity (see [6,13,17]), monotonicity (see [3,4,22]) or bifurcation of critical periodic orbits (see [5,21,23]).

In this paper we consider the two-parameter family of potential differential systems given by

$$X_\mu \quad \begin{cases} \dot{x} = -y, \\ \dot{y} = (1+x)^p - (1+x)^q, \end{cases} \tag{1}$$

where $\mu := (q,p)$ with $p, q \in \mathbb{R}$. This is a well defined analytic differential system on the half plane $\{x > -1\}$. The singular point at the origin is a non-degenerated center if $p > q$ and a hyperbolic saddle if $p < q$. Our goal in this paper is to provide a global study of the qualitative properties of the period function of the center, so we will consider X_μ with $\mu \in \Lambda := \{(q,p) \in \mathbb{R}^2 : p > q\}$. We became interested in this problem because of the previous results by Miyamoto and Yagasaki on the issue. Both authors proved, see [18], that the period function is monotonous when $q = 1$ and $p \in \mathbb{N}$. As it often occurs, they came across the period function when studying the solutions of an elliptic Neumann problem and needed this monotonicity property to prove a bifurcation result. Later Yagasaki improved the result showing in [24] the monotonicity of the period function for $q = 1$ and any $p \in \mathbb{R}$ with $p > 1$. We will prove three main results on the period function of the family $\{X_\mu\}_{\mu\in\Lambda}$. The first one, Theorem A, establishes some regions in the parameter space where the corresponding center has a monotonous period function. This result extends the previous ones by Miyamoto and Yagasaki [18,24]. The second one, Theorem B, deals with the bifurcation of critical periodic orbits from the inner boundary of $\mathscr{P}$, i.e., the center. Finally Theorem C is addressed to the bifurcation of critical periodic orbits from the interior of the period annulus of an isochronous center. These three results, together with the ones that we obtained in [14] concerning the bifurcation of critical periodic orbits from the outer boundary of $\mathscr{P}$, lead us to propose a conjectural bifurcation diagram for the global behaviour of the period function of $\{X_\mu\}_{\mu\in\Lambda}$. We will explain it in detail at the end of this section once we state precisely our main results.

Let us begin with the statement of the monotonicity result. To this end we define

$$\Theta(\mu) := 2p^4 + p^3(3+4q) + p^2(9q^2+9q-1) + p(4q^3+9q^2+2q-3) + (1+q)^2(2q^2-q-1). \tag{2}$$

Then, denoting the light grey region in Fig. 1 by M_I and the dark grey region by M_D, we will prove the following result:

Theorem A. *The period function of the center at the origin of the potential differential system* (1) *is monotonous increasing* (*respectively, decreasing*) *in case that* $\mu \in M_I$ (*respectively,* $\mu \in M_D$).