J. Math. Anal. Appl. 452 (2017) 209–217

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Benedicks' theorem for the Weyl transform

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A R T I C L E I N F O

Article history:
Received 1 July 2016
Available online 1 March 2017
Submitted by M. Peloso

Keywords:
Induced representation
Heisenberg group
Uncertainty principle

A B S T R A C T

If the set of points where a function is nonzero is of finite measure, and its Weyl transform is a finite rank operator, then the function is identically zero.

1. Introduction

That a nonzero function and its Fourier transform cannot both be sharply localized is known as the *Uncertainty Principle* in Harmonic Analysis. There are many different precise formulations of this principle, depending on the way in which localization is quantified. One such is Benedicks' theorem [1]: if $f \in L^1(\mathbb{R})$, and the sets $\{x \in \mathbb{R} \mid f(x) \neq 0\}$ and $\{\xi \in \mathbb{R} \mid \hat{f}(\xi) \neq 0\}$ both have finite Lebesgue measure, then $f \equiv 0$. In this paper, we prove an analog of Benedicks' theorem for the Weyl transform. The Weyl transform is the essence of the group Fourier transform on the Heisenberg group, and also plays a role in the theory of pseudo-differential operators.

Let $\mathcal{H} = L^2(\mathbb{R})$, and $\mathcal{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$. If $f \in L^1(\mathbb{R}^2)$, the *Weyl transform* of f is the operator $W(f) \in \mathcal{B}(\mathcal{H})$ defined by

$$(W(f)\varphi)(t) = \iint f(x, y)e^{\pi i(xy+2yt)}\varphi(t + x)\, dxdy.$$

Our analog of Benedicks' theorem is the following.

Theorem 1.1. *If the set $\{w \in \mathbb{R}^2 \mid f(w) \neq 0\}$ has finite Lebesgue measure and $W(f)$ is a finite rank operator, then $f \equiv 0$.*

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http://dx.doi.org/10.1016/j.jmaa.2017.02.032
0022-247X/

Recall that the Heisenberg group G is the set of triples

$$\{(x, y, z) \mid x, y \in \mathbb{R}, z \in \mathbb{C}, |z| = 1\}$$

with multiplication defined by

$$(x, y, z)(x', y', z') = \left(x + x', y + y', zz' e^{\pi i(xy' - yx')}\right).$$

According to the *Stone–von Neumann Theorem*, there is a unique irreducible unitary representation ρ of G such that

$$\rho(0, 0, z) = zI.$$

The standard realization of this representation is on the Hilbert space $\mathcal{H}$ by the action

$$(\rho(x, y, z)\varphi)(t) = ze^{\pi i(xy+2yt)}\varphi(t + x).$$

Thus, the Weyl transform may be expressed as

$$W(f) = \iint f(x, y)\rho(x, y, 1)\, dxdy, \qquad f \in L^1(\mathbb{R}^2).$$

If X is a trace class operator on $\mathcal{H}$, the modified Fourier–Wigner transform of X is the function $\alpha(X) : \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\alpha(X)(x, y) = \mathrm{tr}(X\rho(x, y, 1)^*).$$

It is well known (see e.g. [2]) that if $f \in L^1(\mathbb{R}^2)$ and $W(f)$ is a trace class operator then $\alpha(W(f)) = f$, and that if X is a trace class operator on $\mathcal{H}$ and $\alpha(X) \in L^1(\mathbb{R}^2)$ then $W(\alpha(X)) = X$. Thus we may reformulate Theorem 1.1 as

Theorem 1.2. *If X is a finite rank operator on $\mathcal{H}$ and the set $\{w \in \mathbb{R}^2 \mid \alpha(X)(w) \neq 0\}$ has finite measure, then $X = 0$.*

Theorem 1.1 was proved in [5] with the hypothesis that f is compactly supported. However, as pointed out by Benedicks in [1], this is a very strong hypothesis, because the support of f is the *closure* of the set $\{w \in \mathbb{R}^2 \mid f(w) \neq 0\}$, and there are open sets of arbitrarily small measure whose closure is all of $\mathbb{R}^2$. The clever argument used in [5] doesn't seem to be susceptible to generalization. Our argument is closer in spirit to Benedicks' original argument, but depends on a deep result of Linnel [3].

2. The double induced realization

Let $\pi : G \to \mathbb{R}^2$ be the projection $\pi(x, y, z) = (x, y)$. Then π is a homomorphism and $\ker(\pi) = Z(G)$, the center of G. Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. For $(x, y) \in \mathbb{R}^2$, let $s(x, y) = (x, y, 1)$. Then s is a section of π, i.e. $\pi \circ s = \mathrm{id}_{\mathbb{R}^2}$, and

$$s(x, y)s(x', y') = (x + x', y + y', e^{\pi i(xy' - yx')}) = \psi((x, y), (x', y'))s(x + x', y + y'),$$

where $\psi((x, y), (x', y')) \in Z(G)$ is defined by

$$\psi((x, y), (x', y')) = (0, 0, e^{\pi i(xy' - yx')}).$$