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Mutual embeddability equivalence relation for rotation algebras

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ABSTRACT

Mutual embeddability is an equivalence relation in C^*-algebras. In this paper, we characterize the classification of rotation algebras in the sense of mutual embeddability.

1. Introduction

Let θ be a real number in $[0, 1)$. Recall that the rotation algebra A_θ is the universal C^*-algebra generated by two unitaries u and v satisfying the Heisenberg relation

$$uv = e^{2\pi i\theta} vu.$$

If $\theta = \frac{q}{p}$ with $(p, q) = 1$, A_θ is a C^*-algebra of type I with spectrum T^2, and the dimension of any irreducible representation of A_θ is p. All tracial states of rational rotation algebra induce the same map from $K_0(A_\theta)$ to $\mathbb{R}$ as shown in [7]. If θ is irrational, A_θ is simple, and has unique tracial state. Moreover, it was shown in [8] that irrational rotation algebras are AT algebras.

Rotation algebra A_θ is $*$-isomorphic to the crossed product C^*-algebra $C(\mathbb{T}) \rtimes_{f_\theta} \mathbb{Z}$, where f_θ is the rotation map on the unit circle $\mathbb{T}$ defined by

$$f_\theta(z) = e^{2\pi i\theta} z, \qquad \text{for any } z \in \mathbb{T}.$$

For convenience, we will not differ the notations A_θ and $C(\mathbb{T}) \rtimes_{f_\theta} \mathbb{Z}$.

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Together with the classification of irrational rotation algebras [15,17] and the classification of rational rotation algebras [3,6,10,13,19,22], all of the rotation algebras are classified.

Theorem 1.1. *Let A_{θ_1} and A_{θ_2} be two rotation algebras, $\theta_1, \theta_2 \in [0,1)$. Then A_{θ_1} is isomorphic to A_{θ_2} if and only if $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$.*

Let A and B be two C^*-algebras. We call A and B are mutually embeddable if there exist an injective $*$-homomorphism $\varphi : A \to B$ and an injective $*$-homomorphism $\psi : B \to A$. Furthermore, we say they are unitally mutually embeddable to each other if those homomorphisms are unital. Obviously, mutual embeddability (unital or not) is an equivalence relation. In this article we study unital mutual embeddability and mutual embeddability relations for rotation algebras.

2. Unitally mutual embeddability equivalence relation for rotation algebras

Proposition 2.1. *Let θ_1 and θ_2 be real numbers in $[0,1)$. Then the rotation algebra A_{θ_1} can be unitally embedded into the rotation algebra A_{θ_2} if and only if there exists an integer k such that $\theta_2 = k\theta_1, mod\ 1$.*

Proof. Let $\iota : A_{\theta_1} \to A_{\theta_2}$ be the embedding $*$-homomorphism. Let $T(A_{\theta_i})$ be the trace space of $A_{\theta_i}, i = 1, 2$. Then ι induces a homomorphism $\iota^* : T(A_{\theta_2}) \to T(A_{\theta_1})$ as following

$$\iota^*(\tau)(a) = \tau(\iota a).$$

Let τ_i be tracial state in $T(A_{\theta_i}), i = 1, 2$ respectively.

$$\iota^*(\tau_2)(1) = \tau_2(\iota(1)) = \tau_2(1) = 1,$$

implies $\iota^*(\tau_2) = \tau_1$ on K-theory level.

There exist Rieffel's projections P_{θ_i} (see [17] for irrational case while see [22] for rational case) such that $\tau_i(P_{\theta_i}) = \theta_i, i = 1, 2$. Recall that

$$\tau_1(\{p, p^2 = p, p^* = p, p \in A_{\theta_1}\}) = \{1 \geq m + \theta_1 n \geq 0, m, n \in \mathbb{Z}\},$$
$$\tau_2(\{p, p^2 = p, p^* = p, p \in A_{\theta_2}\}) = \{1 \geq m + \theta_2 n \geq 0, m, n \in \mathbb{Z}\}.$$

Consequently,

$$\begin{aligned}\theta_1 &= \tau_1(P_{\theta_1}) \\ &= \iota^*(\tau_2)(P_{\theta_1}) \\ &= \tau_2(\iota P_{\theta_1}).\end{aligned}$$

Hence there exist integer m, n s.t.

$$\theta_1 = n\theta_2 + m.$$

Suppose now that $\theta_1 = k\theta_2\ mod\ 1$. Let $u_i, v_i, i = 1, 2$ be generators of rotation algebras. Then the embedding $\iota : A_{\theta_2} \to A_{\theta_1}$ can be defined as

$$\begin{aligned}\iota(u_1) &= u_2^k \\ \iota(v_1) &= v_2.\end{aligned}$$

The proof is now completed. □