J. Math. Anal. Appl. 453 (2017) 1–19

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global attractor for degenerate damped hyperbolic equations ☆

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A R T I C L E I N F O

Article history:
Received 7 November 2016
Available online 30 March 2017
Submitted by Y. Huang

Keywords:
Semilinear degenerate damped hyperbolic equation
Critical growth
Global attractor

A B S T R A C T

The aim of this paper is to consider a class of degenerate damped hyperbolic equations with the critical nonlinearity involving an operator $\mathcal{L}$ that is X-elliptic with respect to a family of vector fields X. We prove the global existence of solutions and characterize their long time behavior. In particular, we show the semigroup generated by the equation has a compact, connected and invariant attractor.

1. Introduction

We study the global existence and longtime behavior of solutions of the problem

$$\begin{cases} \partial_{tt}u(x,t) + \beta u_t(x,t) = \mathcal{L}u(x,t) + f(u(x,t)) & x \in \Omega,\ t > 0, \\ u(x,t) = 0 & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in \Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, where $\mathcal{L}$ is the following degenerate elliptic operator

$$\mathcal{L}u := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}u)$$

and the functions $\{a_{ij}\}$ are measurable in $\mathbb{R}^N$ and $a_{ij} = a_{ji}$.

Let $X := \{X_1, \ldots, X_m\}$ be a family of vector fields in $\mathbb{R}^N$, $X_j = (\alpha_{j1}, \ldots, \alpha_{jN})$, $j = 1, \ldots, m$, where the function α_{jk} is locally continuous in $\mathbb{R}^N$. The notion of X-elliptic operators provides a unifying framework for various types of degenerate elliptic operators. There are two main examples of the operator being

☆ This work was supported by the NSFC Grants (No. 11471148 and No. 11522109).
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http://dx.doi.org/10.1016/j.jmaa.2017.03.077
0022-247X/

considered, one is sub-Laplacians on homogenous Carnot groups (see [3]), the other one is $\triangle_\lambda$-Laplacian (see [12,11]) which is under the additional assumption that the operators are homogenous of degree two with respect to a group of dilations in $\mathbb{R}^N$. X-elliptic operators have been widely studied for several years since it was explicitly introduced in 2000 in [14]. Before that, operators that fall into this class had already been presented in the literature [7,6] and so on. More recently, X-elliptic operators were studied intensely in [8] and [10] (see more references therein), where a maximum principle, a non-homogenous Harnack inequality and a Liouville theorem were obtained.

Our general assumption according to [13] and [10] is that $\mathcal{L}$ is uniformly X-elliptic in the following sense: we identify the vector-valued function X_j with the linear first order partial differential operator

$$X_j = \sum_{k=1}^{N} \alpha_{jk} \partial_{x_k}, \qquad j = 1, \ldots, m.$$

The operator $\mathcal{L}$ is uniformly X-elliptic if there exists a constant $C > 0$ such that

$$\frac{1}{C} \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \leq \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \leq C \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \qquad x, \xi \in \mathbb{R}^N,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$ and

$$\langle X_j(x), \xi \rangle = \sum_{k=1}^{N} \alpha_{jk}(x) \xi_k, \qquad j = 1, \ldots, m.$$

In this paper we show the well-posedness of (1.1) and characterize the long time behavior of solutions using the theory of infinite dimensional dynamical systems. The understanding of asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to study the problem for a dissipative dynamical systems is to consider its global attractor. The existence of global attractors has been proved for a large class of nondegenerate partial equations (see [2,15,19] and references therein). Recently, plenty of papers are devoted to the study of long time behavior of solutions to degenerate parabolic and hyperbolic equations, such as the semilinear heat equation and the semilinear damped wave equation involving the degenerate operator X-elliptic instead of the classical Laplace operator (see [13,12,18]), where the existence of the global attractor and finite fractal dimension were obtained. Compared with the classical problem for the damped hyperbolic equation, the degeneracy decreases the admissible growth of the nonlinearity.

Throughout the paper we assume the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and satisfies the following critical growth condition

$$|f(u) - f(v)| \leq c|u - v|(1 + |u|^\gamma + |v|^\gamma), \quad u, v \in \mathbb{R}, \tag{1.2}$$

for some constant $c \geq 0$, where $\gamma = \frac{q-2}{2}$ and $q = \frac{2Q}{Q-2} > 2$ is determined by the geometry and functional setting associated with the family of vector field X (the explicit form of Q will be recalled in Sect. 2). The existence of the global attractor of our problem (1.1) involving sub-critical growth restriction for the nonlinearity f was established firstly in [13]. Moreover, they show the finite fractal dimension of the global attractor for the generated semigroup and prove convergence of solutions to an equilibrium solutions as time tends to infinity. The aim of the present paper is to extend these results slightly to the case of critical growth condition for the nonlinearity, the more complicated case due to the lack of compactness of the embedding theorems.