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Log-majorization and Lie–Trotter formula for the Cartan barycenter on probability measure spaces

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ABSTRACT

We extend Ando–Hiai's log-majorization for the weighted geometric mean of positive definite matrices into that for the Cartan barycenter in the general setting of probability measures on the Riemannian manifold of positive definite matrices equipped with trace metric. The main key is the settlement of the monotonicity problem of the Cartan barycenteric map on the space of probability measures with finite first moment for the stochastic order induced by the cone. We also derive a version of Lie–Trotter formula and related unitarily invariant norm inequalities for the Cartan barycenter as the main application of log-majorization.

1. Introduction

Let A be an $m \times m$ positive definite matrix with eigenvalues $\lambda_j(A)$, $1 \le j \le m$, arranged in decreasing order, i.e., $\lambda_1(A) \ge \cdots \ge \lambda_m(A)$ with counting multiplicities. The *log-majorization* $A \underset{\log}{\prec} B$ between positive definite matrices A and B is defined if

$$\prod_{i=1}^{k} \lambda_i(A) \le \prod_{i=1}^{k} \lambda_i(B) \quad \text{for } 1 \le k \le m-1, \text{ and } \det A = \det B.$$

The log-majorization gives rise to powerful devices in deriving various norm inequalities and has many important applications in operator means, operator monotone functions, statistical mechanics, quantum information theory, eigenvalue analysis, etc., see, e.g., [4,6,11]. For instance, $A \underset{\log}{\prec} B$ implies $|||A||| \le |||B|||$ for all unitarily invariant norms $|||\cdot|||$.

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As a complementary counterpart of the Golden–Thompson trace inequality, Ando and Hiai [2] established the log-majorization on the matrix geometric mean of two positive definite matrices: for positive definite matrices A, B and $0 \leq \alpha \leq 1$,

$$A^t \#_\alpha B^t \underset{\log}{\prec} (A\#_\alpha B)^t, \qquad t \geq 1,$$

where $A\#_\alpha B := A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2}$, the *α-weighted geometric mean* of A and B. This provides various norm inequalities for unitarily invariant norms via the Lie–Trotter formula $\lim_{t\to 0}(A^t\#_\alpha B^t)^{\frac{1}{t}} = e^{(1-\alpha)\log A+\alpha\log B}$. For instance, $|||(A^t\#_\alpha B^t)^{\frac{1}{t}}|||$ increases to $|||e^{(1-\alpha)\log A+\alpha\log B}|||$ as $r \searrow 0$ for any unitarily invariant norm. Ando–Hiai's log-majorization has many important applications in matrix analysis and inequalities, together with Araki's log-majorization [3] extending the Lieb–Thirring and the Golden–Thompson trace inequalities.

The matrix geometric mean $A\#_\alpha B$, that plays the central role in Ando–Hiai's log-majorization, appears as the unique (up to parametrization) geodesic curve $\alpha \in [0,1] \mapsto A\#_\alpha B$ between A and B on the Riemannian manifold $\mathbb{P}_m$ of positive definite matrices of size m, an important example of Cartan–Hadamard Riemannian manifolds. Alternatively, the geometric mean $A\#_\alpha B$ is the Cartan barycenter of the finitely supported measure $(1-\alpha)\delta_A + \alpha\delta_B$ on $\mathbb{P}_m$, which is defined as the unique minimizer of the least squares problem with respect to the Riemannian distance d (see Section 2 for definition). Indeed, for a general probability measure μ on $\mathbb{P}_m$ with finite first moment, the Cartan barycenter of μ is defined as the unique minimizer as follows:

$$G(\mu) := \underset{Z\in\mathbb{P}_m}{arg\,min} \int_{\mathbb{P}_m} \left[d^2(Z,X) - d^2(Y,X)\right] d\mu(X)$$

(see Section 2 for more details). In particular, when $\mu = \sum_{j=1}^n w_j\delta_{A_j}$ is a discrete probability measure supported on a finite number of $A_1, \ldots, A_n \in \mathbb{P}_m$, the Cartan barycenter $G(\mu)$ is the *Karcher mean* of $A_1, \ldots, A_n$, which has extensively been discussed in these years by many authors as a multivariable extension of the geometric mean (see [7,15,20] and references therein).

The first aim of this paper is to establish the log-majorization (Theorem 4.4) for the Cartan barycenter in the general setting of probability measures in the Wasserstein space $\mathcal{P}^1(\mathbb{P}_m)$, the probability measures on $\mathbb{P}_m$ with finite first moment. In this way, we first establish the monotonicity of the Cartan barycenteric map on $\mathcal{P}^1(\mathbb{P}_m)$ for the stochastic order induced by the cone of positive semidefinite matrices, and then generalize the log-majorization in [2] (as mentioned above) and in [10] (for the Karcher mean of multivariables) to the setting of probability measures. Our second aim is to derive the Lie–Trotter formula (Theorem 5.7) for the Cartan barycenter

$$\lim_{t\to 0} G(\mu^t)^{\frac{1}{t}} = \exp \int_{\mathbb{P}_m} \log A \, d\mu(A)$$

under a certain integrability assumption on μ, where μ^t is the tth power of the measure μ inherited from the matrix powers on $\mathbb{P}_m$. Moreover, to demonstrate the usefulness of our log-majorization, we obtain several unitarily invariant norm inequalities (Corollary 5.8) based on the above Lie–Trotter formula.

The main tools of the paper involve the theory of nonpositively curved metric spaces and techniques from probability measures on metric spaces and the recent combination of the two (see [1,18,19]). Not only are these tools crucial for our developments, but also, we believe, significantly enhance the potential usefulness of the Cartan barycenter of probability measures in matrix analysis and inequalities. They overcome the limitation to the multivariable (finite number of matrices) setting, and provide a new bridge between two different important fields of studies of matrix analysis and probability measure theory on nonpositively curved metric spaces.