J. Math. Anal. Appl. 453 (2017) 221–226

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Connections between centrality and local monotonicity of certain functions on C^*-algebras

Dániel Virosztek [a,b,*,1]

[a] *Department of Analysis, Institute of Mathematics, Budapest University of Technology and Economics, H-1521 Budapest, Hungary*

[b] *MTA-DE "Lendület" Functional Analysis Research Group, Institute of Mathematics, University of Debrecen, H-4002 Debrecen, P.O. Box 400, Hungary*

ARTICLE INFO

Article history:
Received 10 February 2017
Available online 6 April 2017
Submitted by K. Jarosz

Keywords:
C^*-algebra
Centrality
Monotonicity

ABSTRACT

We introduce a quite large class of functions (including the exponential function and the power functions with exponent greater than one), and show that for any element f of this function class, a self-adjoint element a of a C^*-algebra is central if and only if $a \leq b$ implies $f(a) \leq f(b)$. That is, we characterize centrality by local monotonicity of certain functions on C^*-algebras. Numerous former results (including works of *Ogasawara, Pedersen, Wu,* and *Molnár*) are apparent consequences of our result.

1. Introduction

Connections between the commutativity of a C^*-algebra $\mathscr{A}$ and the monotonicity of some functions defined on some subsets of $\mathscr{A}$ have been investigated widely. The first result related to this topic is due to *Ogasawara* who showed in 1955 that a C^*-algebra $\mathscr{A}$ is commutative if and only if the square function is monotone on the positive cone of $\mathscr{A}$ [7]. It was observed later by *Pedersen* that the above statement remains true for any power function with exponent greater than one [8]. *Wu* proved a similar result for the exponential function in 2001 [10]. Ji and Tomiyama showed in 2003 that for any function f which is monotone but not matrix monotone of order 2, a C^*-algebra $\mathscr{A}$ is commutative if and only if f is monotone on the positive cone of $\mathscr{A}$ [2]. The reader is advised to consult the papers [9] and [6] for other closely related results.

Very recently, Molnár proved a local theorem, namely, that a self-adjoint element a of a C^*-algebra $\mathscr{A}$ is central if and only if $a \leq b$ implies $\exp a \leq \exp b$ [5].

* Correspondence to: Department of Analysis, Institute of Mathematics, Budapest University of Technology and Economics, H-1521 Budapest, Hungary.

E-mail address: virosz@math.bme.hu.

URL: http://www.math.bme.hu/~virosz.

[1] The author was supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences and by the National Research, Development and Innovation Office – NKFIH, Grant No. K104206.

http://dx.doi.org/10.1016/j.jmaa.2017.04.008

Motivated by the work of Molnár, we show the following. If $I = (\gamma, \infty)$ is a real interval and f is a continuously differentiable function on I such that the derivative of f is positive, strictly monotone increasing and logarithmically concave, then a self-adjoint element a of a C^*-algebra $\mathscr{A}$ with spectrum in I is central if and only if $a \leq b$ implies $f(a) \leq f(b)$, that is, f is locally monotone at the point a. This result easily implies the results of Ogasawara, Pedersen, Wu, and Molnár.

2. The main theorem

The precise formulation of our main result reads as follows (here and throughout, the symbol $\mathscr{A}_s$ stands for the set of the self-adjoint elements of a C^*-algebra $\mathscr{A}$).

Theorem 1. *Let $I = (\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$ and let $f \in C^1(I)$ be such that*

(i) $f'(x) > 0 \ \ (x \in I)$,
(ii) $x < y \Rightarrow f'(x) < f'(y) \ \ (x, y \in I)$,
(iii) $\log\left(f'\left(tx + (1-t)y\right)\right) \geq t \log f'(x) + (1-t) \log f'(y) \ \ (x, y \in I,\ t \in [0,1])$.

Let $\mathscr{A}$ be a unital C^*-algebra and let $a \in \mathscr{A}$ be a self-adjoint element with $\sigma(a) \subset I$. The followings are equivalent.

(1) *a is central, that is, $ab = ba \ \ (b \in \mathscr{A})$,*
(2) *f is locally monotone at the point a, that is, $a \leq b \Rightarrow f(a) \leq f(b) \ \ (b \in \mathscr{A}_s)$.*

Example. We enumerate the most important examples of intervals and functions satisfying the conditions given in the Theorem:

- $I = (0, \infty),\ f(x) = x^p \ \ (p > 1)$,
- $I = (-\infty, \infty),\ f(x) = e^x$.

3. The proof of the theorem

Notation. If φ and ψ are elements of some Hilbert space $\mathscr{H}$, then the symbol $\varphi \otimes \psi$ denotes the linear map $\mathscr{H} \ni \xi \mapsto \langle \xi, \psi \rangle \varphi \in \mathscr{H}$.

The following proposition is a key step of the proof.

Proposition. *Suppose that $I = (\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$ and $f \in C^1(I)$ satisfies the conditions* (i), (ii) *and* (iii) *given in the Theorem. Let $\mathscr{K}$ be a two-dimensional Hilbert space, let $\{u, v\} \subset \mathscr{K}$ be an orthonormal basis. Let $x, y \in I$ and set $A := xu \otimes u + yv \otimes v$. The followings are equivalent.*

(I) *$x \neq y$,*
(II) *there exist $\lambda, \mu \in \mathbb{C}$ with $|\lambda|^2 + |\mu|^2 = 1$ and $t_0 > 0$ such that using the notation $B = (u + v) \otimes (u + v)$ and $w = \lambda u + \mu v$ we have*

$$\langle f(A)w, w \rangle - \langle f\left(A + t_0 B\right) w, w \rangle > 0.$$

Notation. For any fixed interval $I = (\gamma, \infty)$ and function $f \in C^1(I)$ with the properties (i), (ii) and (iii), and different numbers $x, y \in I$, the above Proposition provides a positive number $\langle f(A)w, w \rangle - \langle f\left(A + t_0 B\right) w, w \rangle$. Let us introduce

$$\delta := \langle f(A)w, w \rangle - \langle f\left(A + t_0 B\right) w, w \rangle .$$