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Realization of functions on the symmetrized bidisc ☆

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ABSTRACT

We prove a realization formula and a model formula for analytic functions with modulus bounded by 1 on the symmetrized bidisc

$$G \stackrel{\text{def}}{=} \{(z+w, zw) : |z| < 1, \ |w| < 1\}.$$

As an application we prove a Pick-type theorem giving a criterion for the existence of such a function satisfying a finite set of interpolation conditions.

1. Introduction

The fascination of the symmetrized bidisc G lies in the fact that much of the classical function theory of the disc $\mathbb{D}$ and bidisc $\mathbb{D}^2$ generalizes in an explicit way to G, but with some surprising twists. The original motivation for the study of G was its connection with the spectral Nevanlinna–Pick problem [3,5], wherefore the emphasis was on analytic maps from the unit disc $\mathbb{D}$ into G. However, in studying such maps one is inevitably drawn into studying maps from G to $\mathbb{D}$; indeed, the duality between these two classes of maps is a central feature of the theory of hyperbolic complex spaces in the sense of Kobayashi [18].

The idea of a *realization formula* for a class of functions has proved potent in both engineering and operator theory. Out of hundreds of papers on this topic in the mathematical literature alone, we mention [20, 15,16,1,9–11,13]. The simplest realization formula provides an elegant connection between function theory (the Schur class of the disc) and contractive operators on Hilbert space. It is as follows.

Let f be an analytic function on $\mathbb{D}$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. There exists a Hilbert space $\mathcal{M}$, a scalar $A \in \mathbb{C}$, vectors $\beta, \gamma \in \mathcal{M}$ and an operator D on $\mathcal{M}$ such that the operator

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$$\begin{bmatrix} A & 1\otimes\beta \\ \gamma\otimes 1 & D \end{bmatrix} \quad \text{is a contraction on } \mathbb{C}\oplus\mathcal{M} \tag{1.1}$$

and, for all $z \in \mathbb{D}$,

$$f(z) = A + \left\langle z(1-Dz)^{-1}\gamma, \beta\right\rangle_{\mathcal{M}}. \tag{1.2}$$

Conversely, any function f on $\mathbb{D}$ expressible in the form (1.1), (1.2) is an analytic function in $\mathbb{D}$ satisfying $|f| \le 1$ on $\mathbb{D}$.

In an earlier paper [7] we gave a realization formula for analytic maps from $\mathbb{D}$ to the closure of G; in this paper we present the dual notion, a realization formula for analytic maps from G to $\mathbb{D}^-$.

For any open set $U \subset \mathbb{C}^d$ the set of analytic functions on U with values in the closed unit disc $\mathbb{D}^-$ is called the *Schur class of* U and is denoted by $\mathscr{S}(U)$.

We shall use superscripts to denote the components of points in $\mathbb{C}^d$.

For any point $s = (s^1, s^2) \in G$ and any contractive linear operator T on a Hilbert space $\mathcal{M}$, we define the operator

$$s_T = (2s^2T - s^1)(2 - s^1T)^{-1} \quad \text{on } \mathcal{M}. \tag{1.3}$$

Note that $|s^1| < 2$ for $s \in G$, and therefore the inverse in equation (1.3) exists.

We shall derive both 'model formulae' and a realization formula for functions in $\mathscr{S}(G)$. The latter is the following.

Theorem 1.1. *Let $\varphi \in \mathscr{S}(G)$. There exist a Hilbert space $\mathcal{M}$ and unitary operators*

$$T \text{ on } \mathcal{M} \quad \text{and} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ on } \mathbb{C}\oplus\mathcal{M} \tag{1.4}$$

such that, for all $s \in G$,

$$\varphi(s) = A + Bs_T(1 - Ds_T)^{-1}C. \tag{1.5}$$

Conversely, any function φ on G expressible by the formula (1.5), where T, A, B, C, D are such that the operators in formula (1.4) are unitary, is an analytic function from G to $\mathbb{D}^-$.

Both instances of the word 'unitary' in the above theorem can validly be replaced by 'contractive'.

The classical realization formula (1.2) is in terms of a single unitary operator (or contraction), whereas our formula for functions in $\mathscr{S}(G)$ requires the *pair* of unitaries (or contractions) (1.4); this is a consequence of the fact that our derivation invokes two separate lurking isometry arguments.

The model formula for functions in $\mathscr{S}(G)$ is derived in Section 2 from the known model formula for $\mathscr{S}(\mathbb{D}^2)$ by a symmetrization argument. The realization formula is then deduced from the model formula in Section 3. A second model formula, involving an integral with respect to a spectral measure, is proved in Section 4. Finally a Pick-type interpolation theorem, giving a solvability criterion for interpolation problems in $\mathscr{S}(G)$, is demonstrated in Section 5. We also give a realization formula for bounded analytic *operator*-valued functions on G. The proof requires only notational changes from that of Theorem 1.1.

This paper is based on a short course of lectures [2] given by the first-named author at the International Centre for the Mathematical Sciences in Edinburgh in 2014.

Two sources for basic facts about the function theory and geometry of G are [17, Chapter 7] and [8, Appendix A].

Many authors have generalized the classical realization formula (1.1) to bounded functions on domains other than the disc. The paper [1] first made it clear that the appropriate class of holomorphic functions