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A global bifurcation result for a class of semipositone elliptic systems ☆

M. Chhetri [a], P. Girg [b,*]

[a] Department of Mathematics and Statistics, The University of North Carolina at Greensboro, Greensboro, NC 27402, United States

[b] Department of Mathematics and NTIS, University of West Bohemia, Univerzitní 8, CZ-30614 Plzeň, Czech Republic

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ABSTRACT

We consider a system of the form

$$\left.\begin{aligned} -\Delta u &= \lambda\left(\theta_1 v^+ + f(v)\right) \quad \text{in} \quad \Omega; \\ -\Delta v &= \lambda\left(\theta_2 u^+ + g(u)\right) \quad \text{in} \quad \Omega; \\ u &= 0 = v \quad \text{on} \quad \partial\Omega, \end{aligned}\right\}$$

where $s^+ \stackrel{\text{def}}{=} \max\{s, 0\}$, θ_1 and θ_2 are fixed positive constants, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with smooth boundary $\partial\Omega$ (a bounded open interval if $N = 1$). The nonlinearities $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions that are bounded from below, sublinear at infinity and have semipositone structure at the origin ($f(0)$, $g(0) < 0$). We show that there are two disjoint unbounded connected components of the solution set and discuss the nodal properties of solutions on these components. Finally, as a consequence of these results, we infer the existence and multiplicity of solutions for λ in a neighborhood containing the simple eigenvalue of the associated eigenvalue problem.

1. Introduction

This paper is devoted to the study of the system

$$\left.\begin{aligned} -\Delta u &= \lambda\left(\theta_1 v^+ + f(v)\right) \quad \text{in} \quad \Omega; \\ -\Delta v &= \lambda\left(\theta_2 u^+ + g(u)\right) \quad \text{in} \quad \Omega; \\ u &= 0 = v \quad \text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

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* Corresponding author.
E-mail addresses: maya@uncg.edu (M. Chhetri), pgirg@kma.zcu.cz (P. Girg).

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where $s^+ \overset{\text{def}}{=} \max\{s,0\}$, θ_1, θ_2 are fixed positive constants, $\lambda \in \mathbb{R}$ is the bifurcation parameter and $\Omega \subset \mathbb{R}^N$ ($N > 1$) is a bounded domain with $C^{2,\eta}$-boundary $\partial\Omega$ for some $\eta \in (0,1)$ (or a bounded open interval if $N = 1$). We set $\tilde{f}(s) \overset{\text{def}}{=} \theta_1 s^+ + f(s)$, $\tilde{g}(s) \overset{\text{def}}{=} \theta_2 s^+ + g(s)$ and make the following assumptions throughout the paper.

(A1) $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, bounded from below and satisfy sublinear condition at infinity, that is,

$$\lim_{|s|\to\infty} \frac{f(s)}{s} = 0 = \lim_{|s|\to\infty} \frac{g(s)}{s}\,;$$

(A2) there exist c, $d > 0$ such that $(s-c)\tilde{f}(s) > 0$ for $s \neq c$ and $(s-d)\tilde{g}(s) > 0$ for $s \neq d$.

We observe that due to (A2), c and d are unique zeros of $\tilde{f}$ and $\tilde{g}$, respectively and that $\tilde{f}(0) < 0$ and $\tilde{g}(0) < 0$ (equivalently, $f(0) < 0$ and $g(0) < 0$). Such nonlinearities are referred in the literature as *semipositone* nonlinearities. It is well documented that study of positive solutions for semipositone problems is more challenging than the so called positone problems (positive and monotone), see e.g. [6,17].

For the positone scalar case, see the survey article [2] where the existence of positive solutions was discussed, among other results, by using the properties of *Positive operators in Banach spaces*. This method does not work if the nonlinearity is negative somewhere. Using different approach based on topological degree and global bifurcation method, authors in [4,11] studied both positone and non-positone case. This technique was utilized further in [3, Theorem 1] to study positive solutions for the semipositone case and in [5, Theorems 3–4] to study more general nonlinearities, that include vanishing type, without assuming any sign condition near the origin. In all these papers, it was shown that there exists an unbounded connected set of solutions containing positive solutions.

More recently for systems, in [12,13,19,20], the existence of positive solution was established for parameters in an interval either to the left or to the right of the simple eigenvalue of the corresponding eigenvalue problem. These papers provide the existence of solutions, but do not provide information on the connectivity of the solution sets. Results in [3,5] were extended to a system of two equations in [8, Theorems 1.2–1.3] without assuming any sign condition on the nonlinear perturbations.

In this paper, by assuming the sign condition (A2), we are able to provide detailed information about the global behavior of some components (see Section 2 for definition) of solution set $\mathscr{S}$ of (1.1). To the best of our knowledge, our results are new for semipositone systems. For the scalar non-positone case, see [4, Theorem C (iii)] where existence of two components containing positive solutions was discussed.

We establish the existence of an unbounded connected component of the solution set emanating from infinity at the simple eigenvalue of the associated nonlinear eigenvalue problem. This component is unbounded in the positive λ-direction and does not intersect the hyperplane $\lambda = 0$. We also show that there exists another connected component of the solution set emanating from the origin which is unbounded in both λ-directions and does not meet the connected component bifurcating from infinity. We discuss the nodal properties of solutions on these components. Finally, as a consequence of these results, we infer the existence and multiplicity of solutions for λ in a neighborhood containing the simple eigenvalue.

In Section 2, we set up notations and discuss nonlinear eigenvalue problem associated with the system (1.1). In Section 3, we state our results and provide examples. In Section 4, we discuss the functional framework to apply bifurcation theory and discuss several preliminary results necessary to prove our results. In Section 5, we establish the nodal properties of solutions on continua of solutions. In Section 6, we prove Theorem 3.1. In Section 7, bifurcation from the trivial solutions is established for the Kelvin transformed problem for (1.1). In Sections 8 and 9, we prove Theorem 3.2 and Theorem 3.3, respectively. In the Appendix, we prove a nonexistence lemma used in Section 5.