J. Math. Anal. Appl. 453 (2017) 338–359

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Subharmonic $P^l$-solutions of first order Hamiltonian systems ☆

Chungen Liu [a,*], Shanshan Tang [b]

[a] School of Mathematics and Information Science, Guangzhou University, Guangzhou, PR China
[b] School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou, PR China

ARTICLE INFO

*Article history:*
Received 19 October 2016
Available online 4 April 2017
Submitted by C.E. Wayne

*Keywords:*
Subharmonic $P^l$-solutions
Maslov $P$-index
Hamiltonian systems

ABSTRACT

In this paper, for any symplectic matrix $P$, the existence of subharmonic $P^l$-solutions of the first order non-autonomous superquadratic Hamiltonian systems is considered. Under the convex condition, the existence of infinitely many geometrically distinct $P^l$-solutions is proved.



## 1. Introduction and main result

For $n \in \mathbb{N}$, we recall that the symplectic group is defined as

$$Sp(2n) \equiv Sp(2n, \mathbb{R}) = \{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^T JM = J\},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the $n \times n$ identity matrix, and $\mathcal{L}(\mathbb{R}^{2n})$ is the space of $2n \times 2n$ real matrices. Without confusion, we shall omit the subindex of the identity matrix.

Let $P \in Sp(2n)$ and $k \in \mathbb{N}$. We say $P$ satisfies $(P)_k$ condition, if $P^k = I$ and $P^m \neq I$ for each integer $m$ with $1 \leq m \leq k-1$.

In this paper, we consider the $(l\tau, P^l)$-boundary problem of first order non-autonomous Hamiltonian systems:

$$\begin{cases} \dot{x} = JH'(t,x), \ \forall x \in \mathbb{R}^{2n}, \ \forall t \in \mathbb{R}, \\ x(l\tau) = P^l x(0), \end{cases} \tag{1.1}$$

☆ The first author was partially supported by the NSF of China (11471170) and the second author was partially supported by the initial Scientific Research Fund of Zhejiang Gongshang University.

* Corresponding author.
*E-mail addresses:* liucg@nankai.edu.cn (C. Liu), ss.tang@foxmail.com (S. Tang).

http://dx.doi.org/10.1016/j.jmaa.2017.03.046
0022-247X/

where $l \in \mathbb{N}$, $\tau > 0$, $P \in Sp(2n)$ satisfying $(P)_k$ condition, $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$, and $H'(t,x)$ is the gradient of $H(t,x)$ with respect to $x \in \mathbb{R}^{2n}$. We call a solution of the problem (1.1) *$P^l$-solution*.

We denote by $(\cdot,\cdot)$, $|\cdot|$ the inner product and norm of $\mathbb{R}^{2n}$ respectively, and by $\mathcal{L}_s(\mathbb{R}^{2n})$ the set of $2n \times 2n$ symmetric real matrices. For $S \in \mathcal{L}_s(\mathbb{R}^{2n})$ we denote its operator norm by $\|S\|$. In this paper we consider the problem (1.1) with the Hamiltonian $H$ given by

$$H(t,x) = \frac{1}{2}(\widehat{B}(t)x, x) + \widehat{H}(t,x),$$

where $\widehat{B}(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n}))$ and $\widehat{H}$ has superquadratic behavior.

The first result for existence of subharmonic periodic solutions of Hamiltonian systems was obtained by P. Rabinowitz in [26] for the case with $\widehat{B}$ being a constant matrix. Since then, many mathematicians made their contributions in this topic. See for example [6,7,16,17,24,30] and the reference therein. In [7], I. Ekeland and H. Hofer proved the existence of infinitely many geometrically distinct periodic solutions of the Hamiltonian systems with $\widehat{B} \equiv 0$ and $H$ being a convex function. In [17], the first author of this paper generalized the result of [7] to the nonconvex case by using the Maslov-type index iteration theory. In [16], the first author and C. Li considered the subharmonic brake solution problem of Hamiltonian systems by using the iteration inequalities of $L$-Maslov type index theory.

For the problem (1.1), we consider the following conditions on $\widehat{H}$ and $\widehat{B}$:

(H1) $\widehat{H} \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ satisfies $\widehat{H}(t+\tau, Px) = \widehat{H}(t,x)$, $\forall t \in \mathbb{R}, x \in \mathbb{R}^{2n}$.
(H2) There exist constants $\mu > 2$ and $R_0 > 0$ such that

$$0 < \mu \widehat{H}(t,x) \le (\widehat{H}'(t,x), x), \quad \forall t \in \mathbb{R},\ |x| \ge R_0.$$

(H3) $\widehat{H}(t,x) = o(|x|^2)$ at $x = 0$.
(H4) $\widehat{H}(t,x) \ge 0$ for all $t \in \mathbb{R}, x \in \mathbb{R}^{2n}$.
(H5) There exist constants $\theta$, $R_1 > 0$ such that

$$|\widehat{H}'(t,x)| \le \theta(\widehat{H}'(t,x), x), \quad \forall t \in \mathbb{R},\ |x| > R_1.$$

(H6) $\widehat{H}''(t,x) > 0$ for all $x \in \mathbb{R}^{2n} \backslash \{0\}$, $t \in \mathbb{R}$.
(H7) $\widehat{B}(t) \in C(\mathbb{R}, \mathcal{L}_s(\mathbb{R}^{2n}))$ is semi-positive definite for all $t \in [0,\tau]$ and satisfies $P^T\widehat{B}(t+\tau)P = \widehat{B}(t)$.

In the main results of this paper we need the following two numbers $\beta$ and $\omega$. $\beta \ge 0$ is defined by

$$\beta = \max_{t \in [0,k\tau]} \|\widehat{B}(t)\|.$$

So there holds $(\widehat{B}(t)z, z) \le \beta |z|^2$, $t \in [0, k\tau]$ and any $z \in \mathbb{R}^{2n}$. In [23] we got a symplectic path $\gamma_P$ satisfying $\gamma_P(0) = I$, $\gamma_P(\tau) = P$ and $-J\gamma_P(t)^{-1}\dot{\gamma}_P(t)$ is negative definite for all $t \in [0,\tau]$. The number $\omega \ge 0$ is defined by

$$\omega = \max_{t \in [0,\tau]} \|\gamma_P(t)^T \widehat{B}(t)\gamma_P(t) + J\gamma_P(t)^{-1}\dot{\gamma}_P(t)\|.$$

**Definition 1.1.** For each positive integer $s$, we define two sequences $\{a_i^s\}_{i=1}^\infty$ and $\{b_j^s\}_{j=1}^\infty$ by

$$a_1^s = s, \quad a_{i+1}^s = (k+1)a_i^s, \quad \forall i \in \mathbb{N}, \tag{1.2}$$

$$b_1^s = s, \quad b_{j+1}^s = (2k+1)b_j^s, \quad \forall j \in \mathbb{N}. \tag{1.3}$$