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On certain multi-variable rational identities derived from the rigidity of signature of manifolds

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ABSTRACT

Song derives certain multi-variable rational identities by studying torus actions on some homogeneous manifolds and applying the Atiyah–Bott–Segal–Singer Lefschetz fixed point theorem. In this paper, we give a direct proof of these rational identities by using the q-Lucas theorem. Moreover, we also give a similar new rational identity.

1. Introduction

Let M be a $4m$ dimensional closed (compact and without boundary) oriented smooth manifold. Let $H^{2m}(M, \mathbb{R})$ denote the middle cohomology group of M with real coefficients. One can introduce a bilinear form

$$B(x, y) = \langle x \cup y, [M] \rangle, \quad x, y \in V = H^{2m}(M, \mathbb{R}).$$

This is a symmetric bilinear form on V. By the Poincare duality, it is non-degenerate. Let p_+ and p_- be the number of positive and negative eigenvalues of B respectively. Define

$$\sigma(M) = p_+ - p_-.$$

When the dimension of M is not divisible by 4, we define $\sigma(M)$ to be 0.

As cohomology is a homotopy invariant of M and $\sigma(M)$ is determined by cohomology, it is a homotopy invariant. Moreover, Thom [15] has shown that $\sigma(M)$ is also a bordism invariant of M. The integer $\sigma(M)$

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is called the *signature* of M. This topological number plays a significant role in the geometry and topology of manifolds. To cite some examples, it was used to construct 4 dimensional topological manifolds, which do not admit smooth structures; it was used to construct Milnor's 7 dimensional exotic sphere, i.e., smooth manifolds that are homeomorphic, but not diffeomorphic, to S^7; it appears in surgery theory, which provides important tools for classification of high-dimensional manifolds.

The signature has profound and rich links to various mathematical theories. The Hirzebruch signature theorem [8] asserts that $\sigma(M)$ is equal to the *L-genus* of M, which is constructed as a polynomial of Pontryagin classes in a way associated to the power series $\frac{x}{\tanh x}$ and therefore relates signature to the theory of characteristic classes. Moreover, by equipping the manifold M with a Riemannian metric g, one finds that $\sigma(M)$ is equal to the index of a first-order elliptic differential operator d_s, called the *signature operator* of the Riemannian manifold (M, g) and therefore relates the signature to differential geometry and analysis on manifolds. The celebrated Atiyah–Singer index theorem asserts that the index of the operator d_s is equal to the L-genus of M, making the Hirzebruch signature theorem a corollary of the Atiyah–Singer index theorem.

In this paper, we study some links of signature to combinatorics. More precisely, we study some combinatorics derived from the signature of certain homogeneous manifolds.

Let $G_k(\mathbb{C}^n)$ denote the Grassmannian manifold of k dimensional complex linear subspaces of $\mathbb{C}^n$. Then $G_k(\mathbb{C}^n)$ is compact and of complex dimension $k(n-k)$ or real dimension $2k(n-k)$. It is a homogeneous manifold, which can be identified with $U(n)/U(k) \times U(n-k)$. The signature of $G_k(\mathbb{C}^n)$ is known to be

$$\sigma(G_k(\mathbb{C}^n)) = \begin{cases} 0 & \text{if } k(n-k) \text{ is odd,} \\ \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & \text{if } k(n-k) \text{ is even.} \end{cases}$$

There are several approaches to compute the signature of $G_k(\mathbb{C}^n)$. The first method is using the Hodge theory (see [8]). The second method is using the Schubert calculus (see [5]). Other methods use various kinds of fixed point formulas and the *rigidity* of the signature. Let M be a compact smooth manifold with an action of a connected Lie group G. Let E and F be two vector bundles on M with the lifted G-actions. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator commuting with the G-action. The *equivariant index* of P is defined to be

$$\mathrm{ind}(P, h) = tr(h|_{KerP}) - tr(h|_{CokerP}), \forall h \in G,$$

which is a class function on G. The elliptic operator P is called *rigid* if $\mathrm{ind}(P, h)$ is a constant independent of $h \in G$. The signature operator d_s is rigid and therefore

$$\sigma(M) = \mathrm{ind}(d_s, \mathrm{id}) = \mathrm{ind}(d_s, h), \forall h \in G.$$

The fixed point theorems then express the right-hand side of the above identity by the data on the fixed point sets of h, and therefore provide tools to compute $\sigma(M)$. In [12], a self-mapping $f : G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$, which is homotopic to the identity map, is constructed and then the Atiyah–Bott Lefschetz fixed point formula [2] is applied to the signature complex (in this case also called the Atiyah–Singer G-signature theorem [3]) to get $\sigma(G_k(\mathbb{C}^n))$. In [9], Hirzebruch and Slodowy consider involutions on homogeneous manifolds and express the signature of the homogeneous manifold as the signature of self-intersection submanifold of the involution when it is homotopic to the identity. This provides another method to compute $\sigma(G_k(\mathbb{C}^n))$. In [13], Song considers the torus action on homogeneous manifold G/H with the torus being the common maximal torus of G and H, and analyzes the fixed points of this action and writes down the signature of G/H by the data