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Perturbed rigidly isochronous centers and their critical periods ☆

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ABSTRACT

This paper investigates the bifurcation of critical periods from a cubic rigidly isochronous center under any small polynomial perturbations of degree n. It proves that for $n = 3$, 4 and 5, there are at most 2 and 4 critical periods induced by periodic orbits of the unperturbed cubic system respectively, and in each case this upper bound is sharp. Moreover, for any $n > 5$, there are at most $\left[\frac{n-1}{2}\right]$ critical periods induced by periodic orbits of the unperturbed cubic system. An example is given to show that the upper bound in the case of $n = 11$ can be reached.

1. Introduction

Consider a two-dimensional differential system

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y), \end{aligned} \tag{1.1}$$

which has a center at the origin. Let $T(\eta)$ be the period of the periodic orbit passing through the point $(\eta, 0)$, called the period function of system (1.1). By convention, a center is called *an isochronous center* if the associated period function is a constant. A center is called a *rigidly isochronous center* if $\dot{\theta} \equiv 1$ in the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ [2]. When $T'(\eta^*) = 0$, $T(\eta^*)$ becomes a critical period.

In the past decades, one has seen that much effort has been dedicated to the study of period function for planar polynomial vector fields, for example, on monotonicity [1,4,8,9,13,17,24], isochronicity [2,20,22], finiteness of critical periods [5,16,18], and local bifurcation of critical periods [6,21,24], which concerns how many critical periods can arise near the center. In recent years, considerable attention is paid to the global

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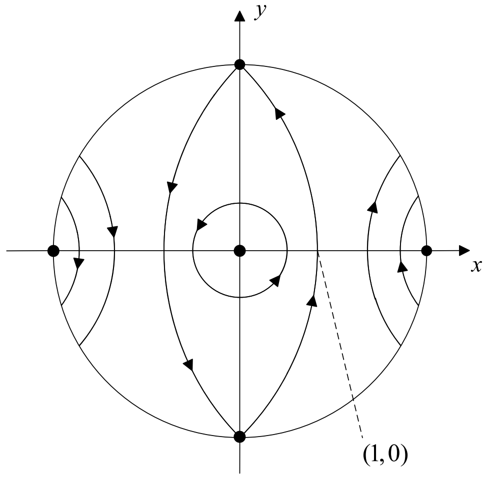

Fig. 1. The phase portrait of system (1.2) in the Poincaré disk.

bifurcation, especially the bifurcation of critical periods from the periodic orbits surrounding the isochronous center under some perturbations [3,7,10–14,19,23]. In our previous work, we considered the bifurcation of critical periods from a rigidly quartic isochronous center [15], and showed that at most one critical period bifurcates from the periodic orbits of the unperturbed quadratic system based on the derived formulas of the qth period bifurcation function for any perturbed isochronous system with a center [19].

It is notable that in [11] some elegant results about the bifurcation of critical periods have been presented when the following system

$$\begin{aligned} \dot{x} &= -y + x^2 y, \\ \dot{y} &= x + xy^2, \end{aligned} \tag{1.2}$$

is perturbed inside $\mathscr{L}_3$, where $\mathscr{L}_3$ is denoted by

$$\begin{aligned} \dot{x} &= -y + P_3(x, y), \\ \dot{y} &= x + Q_3(x, y), \end{aligned}$$

the family of vector fields with a center at the origin, where $P_3(x, y)$ and $Q_3(x, y)$ are homogeneous polynomials of degree 3. In the present paper, we mainly study the bifurcation of critical periods from system (1.2) under any small polynomial perturbation of degree n, and prove that there are at most $2[\frac{n-1}{2}]$ critical periods induced from periodic orbits of system (1.2).

The phase portrait of system (1.2) is depicted in Fig. 1. The origin is a rigidly isochronous center. It is not difficult to find that this system has a first integral $H(x, y) = \frac{x^2+y^2}{1-x^2}$ with an integral factor $\frac{2}{(1-x^2)^2}$. For each $\eta \in (0, 1)$ the orbit through $(\eta, 0)$ lies in the annulus of periodic orbits formed by $\{(x, y)|H(x, y) = c,\ c \in (0, +\infty)\}$, which starts at $(0, 0)$ and terminates at the separatrix passing the infinite degenerate singularity on the equator.

Let us state our main results as follows.

Theorem 1.1. *Assume that for any sufficiently small $|\varepsilon|$, the origin of the following system*

$$\begin{aligned} \dot{x} &= -y + x^2 y + \varepsilon p(x, y), \\ \dot{y} &= x + xy^2 + \varepsilon q(x, y), \end{aligned} \tag{1.3}$$