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Complex dimensions of fractals and meromorphic extensions of fractal zeta functions ☆

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ABSTRACT

We study meromorphic extensions of distance and tube zeta functions, as well as of geometric zeta functions of fractal strings. The distance zeta function $\zeta_A(s) := \int_{A_\delta} d(x,A)^{s-N}\mathrm{d}x$, where $\delta > 0$ is fixed and $d(x,A)$ denotes the Euclidean distance from x to A, has been introduced by the first author in 2009, extending the definition of the zeta function $\zeta_{\mathcal{L}}$ associated with bounded fractal strings $\mathcal{L} = (\ell_j)_{j\geq 1}$ to arbitrary bounded subsets A of the N-dimensional Euclidean space. The abscissa of Lebesgue (i.e., absolute) convergence $D(\zeta_A)$ coincides with $D := \overline{\dim}_B A$, the upper box (or Minkowski) dimension of A. The (visible) complex dimensions of A are the poles of the meromorphic continuation of the fractal zeta function (i.e., the distance or tube zeta function) of A to a suitable connected neighborhood of the "critical line" $\{\mathrm{Re}\, s = D\}$. We establish several meromorphic extension results, assuming some suitable information about the second term of the asymptotic expansion of the tube function $|A_t|$ as $t \to 0^+$, where A_t is the Euclidean t-neighborhood of A. We pay particular attention to a class of Minkowski measurable sets, such that $|A_t| = t^{N-D}(\mathcal{M} + O(t^\gamma))$ as $t \to 0^+$, with $\gamma > 0$, and to a class of Minkowski nonmeasurable sets, such that $|A_t| = t^{N-D}(G(\log t^{-1}) + O(t^\gamma))$ as $t \to 0^+$, where G is a nonconstant periodic function and $\gamma > 0$. In both cases, we show that ζ_A can be meromorphically extended (at least) to the open right half-plane $\{\mathrm{Re}\, s > D - \gamma\}$ and determine the corresponding visible complex dimensions. Furthermore, up to a multiplicative constant, the residue of ζ_A evaluated at $s = D$ is shown to be equal to $\mathcal{M}$ (the Minkowski content of A) and to the mean value of G (the average Minkowski content of A), respectively. Moreover, we construct a class of fractal strings with principal complex dimensions of any prescribed order, as well as with an infinite number of essential singularities on the critical line $\{\mathrm{Re}\, s = D\}$. Finally, using an appropriate quasiperiodic version of the above construction, with infinitely many suitably chosen quasiperiods associated with a two-parameter family of generalized Cantor sets, we construct "maximally-hyperfractal" compact subsets of $\mathbb{R}^N$, for $N \geq 1$ arbitrary. These are compact subsets of $\mathbb{R}^N$ such that the corresponding

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fractal zeta functions have nonremovable singularities at every point of the critical line $\{\operatorname{Re} s = D\}$.

Contents

1. Introduction

1.1. Motivations and goals

This research is a continuation of our work, initiated by the first author and started in [25] (see also [24]), on extending the theory of zeta functions for fractal strings, to fractal sets and arbitrary compact sets in Euclidean spaces. The new zeta function on which it is based has been introduced in 2009 by the first author; see its definition below in Eq. (2.1). We denote this zeta function by ζ_A and refer to it as a "distance zeta function". Here, by a fractal set, we mean any (nonempty) bounded set A of the Euclidean space $\mathbb{R}^N$, with $N \geq 1$. Fractality refers to the fact that the notion of fractal dimension, and in particular, of the upper Minkowski dimension of a bounded set (also called in the literature the upper box dimension, Bouligand dimension, or limit capacity, etc.) is a basic tool in the study of the properties of the associated zeta functions considered in this article, much as is the case in [20] for fractal strings (i.e., when $N = 1$).

More generally, as is shown throughout the higher-dimensional theory of complex dimensions developed in [24–28,22,23,29] and in the present paper, the notion of complex dimensions plays a key role in understanding the geometric oscillations which are intrinsic to fractals. In fact, "fractality" is defined as the existence of a nonreal complex dimension (or else, the existence of a natural boundary for ζ_A, beyond which ζ_A cannot be meromorphically extended). This is the same definition as in [20, §12.1 and §13.4.3], except for the fact that we now have in our possession a general definition and a well developed theory of fractal zeta functions, valid in any dimension; namely, the distance and tube zeta functions (see Definition 2.1 or Definition 3.3, respectively), which extend to arbitrary $N \geq 1$ and any bounded subset of $\mathbb{R}^N$ the usual notion of geometric zeta function of a fractal string (see §2.2). Accordingly, within the present higher-dimensional theory, the visible complex dimensions of a bounded subset A of $\mathbb{R}^N$ are now defined as the poles of the meromorphic continuation (when it exists) of ζ_A to a suitable connected open subset of $\mathbb{C}$; see Definition 2.6.

We next briefly discuss some aspects of the earlier work on fractal strings (or fractal sprays) motivating parts of the present work.