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Centers and limit cycles for a family of Abel equations

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ABSTRACT

Given trigonometric monomials A_1, A_2, A_3, A_4, such that A_1, A_3 have the same signs as $\sin t$, and A_2, A_4 the same signs as $\cos t$, and natural numbers $n, m > 1$, we study the family of Abel equations $x' = (a_1A_1(t) + a_2A_2(t))x^m + (a_3A_3(t) + a_4A_4(t))x^n$, $a_1, a_2, a_3, a_4 \in \mathbb{R}$. The center variety is the set of values a_1, a_2, a_3, a_4 such that the Abel equation has a center (every bounded solution is periodic). We prove that the codimension of the center variety is one or two. Moreover, it is one if and only if $A_1 = A_3$ and $A_2 = A_4$ and it is two if and only if the family has non-trivial limit cycles (different from $x(t) \equiv 0$) for some values of the parameters.

1. Introduction

Smale–Pugh problem consists in determining the maximum number of limit cycles (periodic solutions isolated in the set of periodic solutions) of the equation

$$x' = A(t)x^m + B(t)x^n, \tag{1.1}$$

when A and B belong to given families of T-periodic functions and $m = 2$, $n = 3$, that is for the Abel equation.

This problem was motivated by the second part of Hilbert 16th problem, that consists in determining the maximum number of limit cycles that the family of polynomial planar systems of degree k can have, depending only on this degree k. Many planar systems can be brought into the Abel equation with coefficients being trigonometric polynomials. Examples of these kind of systems are quadratic systems [21], some families of cubic systems [4], systems with homogeneous nonlinearities [11], and rigid systems [16,17].

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A second motivation to study the Smale–Pugh problem is that equation (1.1) with periodic coefficients are used when modeling real-world phenomena (see [5,14,19,26] and references therein).

When $m = 1$, $n = 2$, equation (1.1) is the well-known Riccati equation that has, at most, two limit cycles, see for instance [22]. For the classical Abel equation, $m = 2$ and $n = 3$, A. Lins-Neto [21] proved that there exists no upper bound on the number of limit cycles when A, B belong to the whole family of trigonometric polynomials.

In this way, to tackle the Smale–Pugh problem for the generalized Abel equation, i.e. for any $m, n \in \mathbb{N}$, some restrictions must be imposed on the functions A and B. One kind of restriction that has been considered in literature is the sign invariance of one of the involved functions, or a combination of both. See for instance [3,15,20,24]. In order to bound the number of limit cycles of (1.1), another used restriction has to do with the symmetries of functions A and B. As examples see [6–8].

In [1] a different approach is proposed. Instead of choosing a family of trigonometric polynomials and obtaining an upper bound on the number of limit cycles, in that work the idea is to consider a natural number, N, and obtaining all the families of trigonometric polynomials having at most N limit cycles, only assuming conditions on the degrees of the trigonometric monomials. Concretely, defining $A_l(t) = \sin^{i_l}(t)\cos^{j_l}(t)$, for any fixed N, the authors aim to obtain $i_1, \ldots, i_{k_1+k_2}$, $j_1, \ldots, j_{k_1+k_2}$, such that the following Abel equation

$$x' = \left(\sum_{l=1}^{k_1} a_l A_l(t)\right) x^m + \left(\sum_{l=k_1+1}^{k_1+k_2} a_l A_l(t)\right) x^n, \tag{1.2}$$

has at most N limit cycles for every $a_1, \ldots, a_{k_1+k_2} \in \mathbb{R}$, and there are equations with N limit cycles for some $a_1, \ldots, a_{k_1+k_2} \in \mathbb{R}$.

The case $N = 0$ is related with the center problem, and in this setting is not difficult to be solved (see [1]). The polynomial case has been studied in [9,10], where conditions on the degrees of A and B are given in order to have a center at the origin.

The first interesting case is $N = 1$, or in other words, identify the families of Abel equations (1.1) such that (globally) perturbing a_l, b_l the unique possible limit cycle is the trivial one $x(t) \equiv 0$. This is equivalent to identify all the families such that perturbing a_l, b_l one obtains non-trivial limit cycles.

The case $N = 1$ and $k_2 = 1$, has been solved in [2]. Now we want to deal with $N = 1$ and $k_2 > 1$. Note that if for some election of $i_1, \ldots, i_{k_1+k_2}$, $j_1, \ldots, j_{k_1+k_2}$ we prove that $N = 1$, then removing some of the monomials in (1.2), we have $N \leq 1$. Using [1, Theorem 4.1], we only need to solve the problem for a few families of equations remaining. See Remark 1.2 below for the details.

In this paper, we consider one of these families,

$$x' = (a_1 A_1(t) + a_2 A_2(t))x^m + (a_3 A_3(t) + a_4 A_4(t))x^n = A(t)x^m + B(t)x^n, \tag{1.3}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, $n > m > 1$ are fixed integers, and A_1, A_2, A_3, A_4 are fixed trigonometric monomials such that

$$A_1(t), A_3(t) \in \mathcal{S}, \quad A_2(t), A_4(t) \in \mathcal{C},$$

with

$$\mathcal{S} = \{\sin^i(t)\cos^j(t) \colon i \text{ odd}, \ j \text{ even}\}, \quad \mathcal{C} = \{\sin^i(t)\cos^j(t) \colon i \text{ even}, \ j \text{ odd}\}.$$

Note that since $n, m > 1$, equation (1.3) always has the solution $x(t) \equiv 0$ (trivial solution). Moreover, the trivial solution is a limit cycle or every bounded solution is periodic. In this second case we say that (1.3)