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On distribution of local dimensions of doubling measures on Euclidean space ☆

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ABSTRACT

We prove that for every nonnegative, increasing and right continuous function g on $[0, n]$ with $g(n) = 1$ and $g(\epsilon) = 0$ for some $\epsilon \in (0, n)$ there exists a doubling probability measure μ on $[0, 1]^n$ such that the distribution G_μ of the lower local dimension of μ is g.

1. Introduction

Let μ be a finite Borel measure on $[0, 1]^n$. We define the lower and upper local dimension of μ at $x \in [0, 1]^n$ by

$$\underline{\dim}_{loc}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r}$$

and

$$\overline{\dim}_{loc}\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x, r))}{\log r},$$

here and after $B(x, r)$ denotes the open ball with center x and radius $r > 0$. And we say that the local dimension exists at x if these are equal, writing $\dim_{loc} \mu(x)$ for the common value. Note that $\dim_{loc} \mu(x) = 0$ if $\mu\{x\} > 0$, and $\dim_{loc} \mu(x) = \infty$ if $\mu(B(x, r)) = 0$ for some $r > 0$.

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It is well known that the lower local dimension is closely related to the Hausdorff dimension by the following Theorem A (see Proposition 2.3 in [1]).

Theorem A. *Let μ be a Borel probability measure on $[0,1]^n$. Then*
(a) $\dim_H E \geq s$, *if* $E \subset \{x \in [0,1]^n : \underline{\dim}_{loc}\mu(x) \geq s\}$ *and* $\mu(E) > 0$.
(b) $\dim_H\{x \in [0,1]^n : \underline{\dim}_{loc}\mu(x) \leq s\} \leq s$.
(c) $\dim_H E = s$, *if* $E = \{x \in [0,1]^n : \underline{\dim}_{loc}\mu(x) = s\})$ *and* $\mu(E) > 0$.

Theorem A implies that $\underline{\dim}_{loc}\mu(x) \in [0,n]$ for μ-almost all $x \in [0,1]^n$. Define

$$G_\mu(s) = \mu\{x \in [0,1]^n : \underline{\dim}_{loc}\mu(x) \leq s\},\ s \in [0,n]. \tag{1}$$

Then $G_\mu(s)$ is nonnegative, increasing, and right continuous with $G_\mu(n) = 1$. The function $G_\mu(s)$ is called the distribution of the lower local dimension of μ. It is well known that for every nonnegative, increasing and right continuous function g on $[0,n]$ with $g(n) = 1$ there exists a Borel probability measure μ on $[0,1]^n$ such that $G_\mu = g$; see for example Proposition 10.12 in [1]. That's to say, the distribution of the lower local dimension of μ can be arbitrary.

In this paper we study the distribution of the local dimension of a doubling measure on $[0,1]^n$. Recall that a Borel measure μ on a metric space X is called doubling, if there is a constant $C \geq 1$ such that

$$0 < \mu(B(x,2r)) \leq C\mu(B(x,r)) < +\infty \tag{2}$$

for every ball $B(x,r)$ in X. In this case, μ is said to be C-doubling. It is known that every complete doubling metric space carries a doubling measure; see Volberg–Konyagin [8] and Luukkainen–Saksman [6]. Also, the singularity of doubling measures was studied. Kaufman–Wu [4] proved that for every compact doubling metric space X with no isolated points and for every doubling measure μ on X, there exists a doubling measure on X singular with μ. Wu [11] showed that for every $\alpha > 0$ a doubling measure on a compact doubling metric space can be supported on a set of Hausdorff dimension at most α. The same result holds for packing dimension, see Käenmäki–Rajala–Suomala [3]. Our main result is the following theorem.

Theorem 1. *The distribution G_μ of the lower local dimension of a doubling probability measure μ on $[0,1]^n$ is a nonnegative, increasing and right continuous function on $[0,n]$ with $G_\mu(n) = 1$ and $G_\mu(\epsilon) = 0$ for some $\epsilon \in (0,n)$. Conversely, for every nonnegative, increasing and right continuous function g on $[0,n]$ with $g(n) = 1$ and $g(\epsilon) = 0$ for some $\epsilon \in (0,n)$ there exists a doubling probability measure μ on $[0,1]^n$ such that $G_\mu = g$.*

Theorem 1 together with Theorem A shows that doubling measures on $[0,1]^n$ can be of any singular structure permitted by the assumption of Theorem 1. For the other works on doubling measures we refer to [5,7,9,10].

2. Preliminary

The proof of Theorem 1 is based on the following two lemmas.

Lemma 1. *Let μ be a finite Borel measure on $[0,1]^n$ and $s \in [0,n]$. Then*

$$\mu(\underline{E}_{\leq s}) = \sup\{\mu(E) : E \subset [0,1]^n \text{ and } \dim_H E \leq s\}, \tag{3}$$

where $\underline{E}_{\leq s} = \{x \in [0,1]^n : \underline{\dim}_{loc}\mu(x) \leq s\}$. In other words, the set $\underline{E}_{\leq s}$ is of maximal μ-measure among those of Hausdorff dimension at most s.