Doctopic: Optimization and Control ARTICLE IN PRESS YJMAA:21303

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weak boundary penalization for Dirichlet boundary control problems governed by elliptic equations

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A R T I C L E I N F O

Article history:
Received 8 December 2016
Available online xxxx
Submitted by H. Zwart

Keywords:
Dirichlet boundary control
Weak boundary penalization
Finite element method
A priori error estimate

A B S T R A C T

This paper concerns the finite element approximation of Dirichlet boundary control problems governed by elliptic equations. Different from the existing literatures, in which standard finite element method, mixed finite element method or Robin penalization method are used to deal with the underlying problems, we adopt an alternative penalization approach introduced by Nitsche called weak boundary penalization. Compared with the above methods, our discrete scheme not only keeps consistency and avoids penalization error, but also can be analyzed and computed conveniently as Neumann boundary control problems. Based on the weak boundary penalization method, we establish a finite element approximation to the Dirichlet boundary control problems and derive the a priori error estimates for the control, state and adjoint state. Numerical experiments are provided to confirm our theoretical results.

1. Introduction

Finite element approximation of optimal control problems plays an important role in the numerical methods of these problems. There have been extensive theoretical and numerical studies on finite element approximation of various control problems. For instance, the error analysis for control problems governed by linear elliptic equations has been established in [16,18]. Some progress in this area has been summarized in [21,25]. However, these work are mainly concentrated on distributed control problems. Boundary control is a kind of very important problems and finds many applications in especially fluid dynamics. Depending on different boundary conditions where the control acts as, it can be classified into the Neumann-type, Robin-type and Dirichlet-type. Casas and Mateos present numerical analysis for Neumann boundary control

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http://dx.doi.org/10.1016/j.jmaa.2017.04.016
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of elliptic equations in [8,11]. These results also hold for the Robin-type (see [21]). Additionally, Liu and Yan [24] obtained the a posteriori error estimates for Neumann boundary control problems.

In contrast with the Neumann-type, the key point and difficulty in handling the Dirichlet boundary control problems are the choice of appropriate control space. In this paper, we study the finite element method for the following Dirichlet boundary control problem

$$\min_{u\in K} \quad J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 dx + \frac{\alpha}{2}\int_\Gamma u^2 ds \tag{1.1}$$

subject to

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = u & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded convex domain with Lipschitz boundary $\Gamma = \partial\Omega$, $y_d \in L^s(\Omega)$ and $s > 2$, $\alpha > 0$ is a regularization parameter. We denote the set of admissible control by

$$K = \big\{ v \in L^2(\Gamma) : a \leqslant v(x) \leqslant b \quad \text{a.e. } x \in \Gamma \big\},$$

where $a < b$ are constants.

Given $u \in L^2(\Gamma)$ and $f \in L^2(\Omega)$, the Dirichlet boundary value problem (1.2) admits a unique solution $y \in H^{\frac{1}{2}}(\Omega)$ (see [4,9]) in the sense that

$$-\int_\Omega y\Delta w dx = \int_\Omega f w dx - \int_\Gamma u \frac{\partial w}{\partial n} ds \quad \forall\, w \in H_0^1(\Omega) \cap H^2(\Omega), \tag{1.3}$$

where $n \in \mathbb{R}^2$ is the unit outward normal to Γ. Using standard arguments (see [23]), it can be proved that the problem (1.1)–(1.2) admits a unique solution $(y,u) \in H^{\frac{1}{2}}(\Omega) \times L^2(\Gamma)$ which can be characterized by the first order optimality conditions

$$\int_\Gamma (\alpha u - \frac{\partial p}{\partial n})(v-u) ds \geqslant 0 \quad \forall\, v \in K, \tag{1.4}$$

or

$$u(x) = \mathrm{Proj}_K\Big(\frac{1}{\alpha}\frac{\partial p}{\partial n}\Big) = \max\Big(a, \min\Big(b, \frac{1}{\alpha}\frac{\partial p}{\partial n}\Big)\Big), \tag{1.5}$$

where p, the adjoint variable, satisfies the equation

$$\begin{cases} -\Delta p = y - y_d & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma. \end{cases} \tag{1.6}$$

There are several literatures contributed to the finite element methods for Dirichlet boundary control problems. Casas and Raymond [9] considered the semilinear Dirichlet control problems imposed on convex polygonal domains, and derived the error estimates $||u-u_h||_{0,\Gamma} \leqslant Ch^{1-\frac{1}{s}}$ for some $s > 2$, which is consistent with the $W^{1-\frac{1}{s},s}(\Gamma)$-regularity of the control variable. May et al. [27] considered unconstrained problem imposed on convex polygonal domains and derived the optimal result $||y-y_h||_{0,\Omega} \leqslant Ch^{\frac{3}{2}-\frac{1}{s}}$ and $||u-u_h||_{0,\Gamma} \leqslant Ch^{1-\frac{1}{s}}$. Deckelnick et al. [15] studied the above control problem imposed on smooth domains, they obtained the error estimate