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Solvability of a parabolic problem with non-smooth data

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A B S T R A C T

We study a weak solvability of one linear parabolic problem with a summable right hand side and non-smooth coefficients. We consider this problem as appropriate operator equation in some Banach space. Methods of the semigroups theory, fractional powers of operators and integral operators for solvability of this operator equation are used.

1. Introduction

We are interested in the question of the solvability in a weak sense of the following initial–boundary value problem

$$\frac{\partial v(t,x)}{\partial t} + \sum_{i=1}^{n} a_i(t,x)\frac{\partial v(t,x)}{\partial x_i} - \chi\Delta v(t,x) = f(t,x),\ (t,x) \in Q_T; \tag{1.1}$$

$$v(0,x) = v^0(x),\ x \in \Omega; \quad v(t,x) = 0,\ 0 \le t \le T,\ x \in \partial\Omega. \tag{1.2}$$

Here $\Omega \subset R^n$, $n = 2,3$, is a bounded domain with the boundary $\partial\Omega \subset C^2$, $Q_T = [0,T] \times \Omega$, $\chi > 0$, $a(t,x) = (a_1(t,x),\dots,a_n(t,x)) \in L_2(0,T;W_2^1(\Omega)^n)$ is divergence free, $f \in L_1(0,T;L_1(\Omega))$ and v^0 are given and $v(t,x)$ is unknown.

Such parabolic system arises in different problems of mathematical physics. For example, below we mention the application of this problem in continuum mechanics.

It is well known that the system of equations describing thermoviscoelastic incompressible continuum dynamics consists of a motion equations, a rheological relation (detailing continuum, see e.g. [22]) between

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the deviator of the stress tensor and the strain rate tensor and an energy balance equation (see e.g. [4,31]). One of the usually used approaches to study the solvability of this system in a weak sense is as follows (see e.g. [29–34]). First, for fixed temperature v one finds the velocity $a \in L_2(0,T;W_2^1(\Omega))$ and the deviator of the stress tensor $\sigma \in L_2(0,T;L_2(\Omega))$ from the system consisting of the motion equation and rheological relation. Then a and σ are inserted in the energy balance equation. This yields the equation with respect to the temperature v. In this way the question of the solvability in a weak sense of thermoviscoelastic continuum dynamics system is reduced to the study of the solvability of an initial–boundary value parabolic problem corresponding to the energy balance equation with non-smooth coefficients and a right hand side part from L_1.

There is a wide literature on parabolic and elliptic problems with right hand side from L_p, $1 < p < +\infty$ (see e.g. [2,14,15,24,25] and the bibliography therein).

During last time parabolic and elliptic problems with right hand side from L_1 have been intensively studied (see e.g. [1,3,5–12,17–19,28] and the bibliography therein). In these works difficulties caused by right hand side from L_1 are overcome in different ways. In [5–7] the study is based on the use of the concept of renormalized (or entropy) solutions. The concept of duality and weak solutions was used in [12,18,19,28]. In [1,8–10] the approach based on the theory of analytic semigroups in $L_1(\Omega)$ and the theory of interpolation spaces was used.

However, results of these works cannot be applied to our problem (1.1)–(1.2) due to the fact that coefficients $a_i(t,x) \in L_2(0,T;W_2^1(\Omega))$ in (1.1) are non-smooth.

So, our goal is to investigate problem (1.1)–(1.2) for $a_i(t,x) \in L_2(0,T;W_2^1(\Omega))$.

Our approach to the study of the solvability of problem (1.1)–(1.2) is based on the application of the fractional powers of positive operators theory, some properties of integral operators of potential type and the theory of analytic semigroups in $L_p(\Omega)$, $p > 1$.

The regularization of the original equation by application of a negative fractional power of a positive operator to (1.1) yields the better anisotropic summability of the right hand side of problem (1.1)–(1.2). This allows us to establish necessary anisotropic L_p estimates of solutions to problem (1.1)–(1.2) depending on the space dimension. These estimates are essential, for example, for the proof of the solvability (in a weak sense) of various thermoviscoelasticity equations (see [31]).

The paper is organized as follows. In Section 2 necessary notations and auxiliary facts are given and the main result (Theorem 2.1) is formulated. Section 3 is devoted to an operator reformulation of the main problem. In Section 4 the proof of the main result (in form of Theorem 3.1) is given. This Section is divided into a number of Subsections devoted to the proof of necessary auxiliary results and the direct proof of main Theorem 2.1.

2. Notations and the formulation of the main result

We use Sobolev spaces $L_p(\Omega)$, $W_p^l(\Omega)$, $L_p(Q_T)$, $W_p^{k,m}(Q_T)$, $1 \le p < +\infty$, of functions defined on Ω and Q_T with values in R^1. Norms in these spaces for $p = 2$ are denoted by $|\cdot|_0$, $|\cdot|_l$, $\|\cdot\|_0$, $\|\cdot\|_{k,m}$, respectively. Denote by $C_0^\infty(\Omega) \subset L_p(\Omega)$ the set of infinitely differentiable compactly supported functions on Ω, by $\overset{\circ}{W}{}_p^m(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm of space $W_p^m(\Omega)$ $(m > 0)$ and $W_{p,0}^m(\Omega) = W_p^m(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$, $m > 1/p$.

Next, $W_p^{-m}(\Omega) = (\overset{\circ}{W}{}_q^m(\Omega))'$, $m > 0$, $q = p/(p-1)$, $1 < p < +\infty$, where the sign $'$ denotes the conjugate space. Symbols H and V denote the completion of $\mathcal{V} = \{u : u \in C_0^\infty(\Omega)^n, \operatorname{div} u = 0\}$ with respect to the norms of spaces $L_2(\Omega)^n$ and $W_2^1(\Omega)^n$ $(n = 2,3)$, respectively (see [26], Sec. 1.4).

Let us denote by A the operator $Au = -\chi\Delta u$, $\chi > 0$, with the domain $D(A) = W_{p,0}^2(\Omega)$, which acts in $L_p(\Omega)$, $1 < p < +\infty$. The operator $A : W_{p,0}^2(\Omega) \to L_p(\Omega)$ is the generator of the analytic semigroup $S(t) = \exp(-tA)$, $t \ge 0$. Negative fractional powers $A^{-\alpha}$, $\alpha > 0$, of operator A are defined by the formula