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Large time decay of solutions to the Boussinesq system with fractional dissipation

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A B S T R A C T

In this paper we extend the results of Brandolese and Schonbek [2] to the Boussinesq system with fractional dissipation. Let Λ^α and Λ^β represent the fractional Laplacian dissipation in the velocity and the temperature equations, respectively. We show that if the initial data $(u_0, \theta_0) \in L^2_\sigma \times (L^1 \cap L^2)$, then $\|\theta(t)\| \le C(1+t)^{-\frac{d}{2\beta}}$, and $\|u(t)\| \le C(1+t)^{\max\{0,1-\frac{d}{2\beta}\}}$ if $\beta \ne \frac{d}{2}$, $\|u(t)\| \le Cln(2+t)$ if $\beta = \frac{d}{2}$; if we additionally assume $\int \theta_0 = 0$ and $\theta_0 \in L^1_1$, then $\|\theta(t)\| \le C(1+t)^{-\frac{d+2}{2\beta}}$ and $\|u(t)\| \to 0$ as $t \to \infty$. Comparing [2], we don't need to assume that $\|\theta_0\|_1$ is sufficiently small when $\beta \in (0, \frac{d+1}{2})$.

1. Introduction

In this paper we consider the following incompressible Boussinesq system with fractional dissipation

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu\Lambda^\alpha u = -\nabla p + \theta e_d, & x\in\mathbb{R}^d,\ t>0,\\ \partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^\beta\theta = 0, & x\in\mathbb{R}^d,\ t>0,\\ \nabla\cdot u = 0, & x\in\mathbb{R}^d,\ t>0,\\ u(x,0)=u_0(x), \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the velocity, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, $\nu > 0$, $\kappa > 0$, $\alpha \in (0,2]$ and $\beta \in (0,2]$ are real parameters, and $e_d = (0,...,0,1)$, $d \ge 2$. $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator and Λ^α can be defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi),$$

where

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$$\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx.$$

By recalling the unknowns, without loss of generality, we take $\nu = \kappa = 1$.

Definition 1.1. (u, θ) is called a weak solution of (1.1) if $u \in L^\infty_{loc}(0, \infty; L^2_\sigma(\mathbb{R}^d)) \cap L^2_{loc}(0, \infty; H^{\frac{\alpha}{2}}(\mathbb{R}^d))$ and $\theta \in L^\infty_{loc}(0, \infty; L^2(\mathbb{R}^d)) \cap L^2_{loc}(0, \infty; H^{\frac{\beta}{2}}(\mathbb{R}^d))$ satisfy

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}^d} u \cdot \partial_t \phi dxdt + \int_0^\infty \int_{\mathbb{R}^d} \Lambda^{\frac{\alpha}{2}} u \cdot \Lambda^{\frac{\alpha}{2}} \phi dxdt + \int_0^\infty \int_{\mathbb{R}^d} u \cdot \nabla u \cdot \phi dxdt \\ &= \int_0^\infty \int_{\mathbb{R}^d} u_0 \cdot \phi(0) dx + \int_0^\infty \int_{\mathbb{R}^d} \theta e_d \cdot \phi dxdt \quad \text{for all } \phi \in C_0^\infty([0,\infty); C^\infty_{0,\sigma}(\mathbb{R}^d)), \end{aligned}$$

and

$$\begin{aligned} &-\int_0^\infty \int_{\mathbb{R}^d} \theta \cdot \partial_t \psi dxdt + \int_0^\infty \int_{\mathbb{R}^d} \Lambda^{\frac{\beta}{2}} \theta \cdot \Lambda^{\frac{\beta}{2}} \psi dxdt + \int_0^\infty \int_{\mathbb{R}^d} u \cdot \nabla \theta \cdot \phi dxdt \\ &= \int_0^\infty \int_{\mathbb{R}^d} \theta_0 \cdot \phi(0) dx \quad \text{for all } \psi \in C_0^\infty([0,\infty); C_0^\infty(\mathbb{R}^d)), \end{aligned}$$

where $(u_0, \theta_0) \in L^2_\sigma(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$.

When $\alpha = 2$, $\beta = 2$, the system (1.1) becomes the standard Boussinesq equations with Laplacian dissipation. The standard Boussinesq equation is widely used to model the dynamics of the ocean or the atmosphere, see e.g. [13]. It arises from the density dependent incompressible Navier–Stokes equations by using the so-called Boussinesq approximation, which consists in neglecting the density dependence in all the terms but the one involving the gravity. The standard Boussinesq system had lately received significant attention in mathematical fluid dynamics due to its connection to incompressible Navier–Stokes equations. The global existence of weak solutions, or strong solutions in the cases of small data had been studied by several authors, see, e.g., [1,4,7,8,10,14]. The Boussinesq system with fractional dissipation also had been considered by many mathematicians, see [11,18] and the references therein. They obtained the global regularity for (1.1) with α and β in some intervals.

Another interesting question is the large time behavior of the incompressible flows. Many interesting and important results on the decay properties had been achieved for Navier–Stokes flows, see e.g. [12,15–17]. In [9], Guo and Yuan considered the decay of the weak solutions to the standard Boussinesq system. They claimed that if $(u_0, \theta_0) \in (L^1 \cap L^2_\sigma) \times (L^1 \cap L^2)$, then

$$\|\theta(t)\| \le C(1+t)^{-\frac{5}{4}}$$

and

$$\|u(t)\| \to 0, \quad \text{as } t \to \infty.$$

Recently, by using Fourier transform and a straightforward adaptation of Caffarelli, Kohn and Nirenberg's method in [3], Brandolese and Schonbek [2] considered the decay properties of weak and strong solutions

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