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Spectral property of certain fractal measures ☆

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A B S T R A C T

Let $\{0, a_j, b_j\} = \{0, 1, 2\}(mod\ 3)$ be a sequence of digit sets in $\mathbb{Z}$, and let $\{N_j = 3r_j\}$ be a sequence of integers bigger than 1. We call $\{f_{j,d}(x) = N_j^{-1}(x + d) : d \in \{0, a_j, b_j\}\}_{j=0}^{\infty}$ a Moran iterated function system, which is a generalization of an IFS. We prove that the associated Moran measure is spectral.

1. Introduction

We say that a compactly supported probability measure μ is a spectral measure if there exists a set of complex exponentials $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda}$ such that it is an orthonormal basis of $L^2(\mu)$. If such Λ exists, it is called a spectrum of μ. We also say a set Ω is a spectral set if $\chi_\Omega dx$ is a spectral measure. The study of spectral sets was first initiated from B. Fuglede in 1974 [10]. He proposed a reasonable conjecture on spectral sets:

Fuglede's Conjecture. $\Omega \subset \mathbb{R}^s$ is a spectral set if and only if Ω is a translational tile.

The problem of spectral measures is exciting when we consider fractal measures. Jorgensen and Pedersen [13] showed that the standard Cantor measure is a spectral measure if the contraction is $\frac{1}{2n}$, while there are at most two orthogonal exponentials when the contraction is $\frac{1}{2n+1}$. Following this discovery, more spectral fractal measures were found [1–6,8,7,9,11,12,14–16]. In particular, An and He [1] constructed a class of Moran spectral measures. Motivated by their ideas, we will focus on certain Moran measures.

Two finite sets $\mathcal{A} = \{a_j\}$ and $\mathcal{S} = \{s_j\}$ of cardinality q in $\mathbb{R}$ form a compatible pair, following the terminology of [16], if the matrix $M = [\frac{1}{\sqrt{q}}e(a_j s_k)]$ is a unitary matrix. In other words $(\delta_{\mathcal{A}}, \mathcal{S})$ is a spectral pair, where

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$$\delta_{\mathcal{A}} := \sum_{a\in\mathcal{A}} \frac{1}{q}\delta_a.$$

A compatible tower is a sequence (finite or infinite) of compatible pairs

$$\{B_0, L_0\}, \{B_1, L_1\}, \{B_2, L_2\}, \cdots$$

with $B_j \subset M_j^{-1}\mathbb{Z}^n$ and $L_j \subset \mathbb{Z}^n$, and matrices $R_j \in GL(n, M_{j-1}\mathbb{Z}^n)$ for $j \geq 1$.

Let $\{N_j\}_{j=0}^{\infty}$ be a sequence of integers with all $N_j \geq 2$ and let $\{D_j\}_{j=0}^{\infty}$ be a sequence of digit sets with $0 \in D_j \subset \mathbb{N}$ for each $j \geq 0$. We say $\{f_{j,d}(x) = N_j^{-1}(x+d) : d \in D_j\}_{j=0}^{\infty}$ is a Moran iterated function system, which is a generalization of an IFS. If $\sup\{d : d \in D_j, j \geq 0\} < \infty$, Strichartz [17] proved that there exists a compact set T and a Borel probability measure μ_T supported on T. Moreover,

$$T = \sum_{j=0}^{\infty}(N_0N_1\cdots N_j)^{-1}D_j = \left\{\sum_{j=0}^{\infty}(N_0N_1\cdots N_j)^{-1}d_j : d_j \in D_j, j \geq 0\right\}$$

and

$$\mu_T = \delta_{N_0^{-1}D_0} * \delta_{(N_0N_1)^{-1}D_1} * \cdots * \delta_{(N_0N_1\cdots N_j)^{-1}D_j} * \cdots,$$

where $*$ is the convolution sign.

Let $\mathcal{N} = \{N_j : N_j = 3r_j, r_j \in \mathbb{Z}^+, j = 0, 1, 2, \cdots\}$ and $\mathcal{D} = \{D_j : D_j = \{0, a_j, b_j\} \subset \mathbb{N}\}$ where $\sup\{d : d \in D_j, j \geq 0\} < \infty$ and $a_j \in 3\mathbb{Z}+1, b_j \in 3\mathbb{Z}+2, j = 0, 1, 2, \cdots$. We use $\mu_{\mathcal{N},\mathcal{D}}$ to denote the corresponding Moran measure.

Theorem 1.1. *$\mu_{\mathcal{N},\mathcal{D}}$ is a spectral measure with a spectrum*

$$\Lambda = r_0L + r_0N_1L + \cdots + r_0N_1\cdots N_kL + \cdots$$

where $L = \{-1, 0, 1\}$ and each element of Λ is a finite sum.

Remark 1.2.

(1) Theorem 2.8 in [16] indicates that, to obtain uniform control in the use of Dominated Convergence Theorem, expanding matrices $\{R_j\}$ must be chosen from a finite set of expanding matrices. However, $\{N_j\}$ in Theorem 1.1 can be chosen from an infinite set of integers.

(2) If $N_j = 3$, $\mathcal{D}_j = \{0, 1, 2\}$ for all $j \in \mathbb{N}$, then $\mu_{\mathcal{N},\mathcal{D}} = \chi_{[0,1]}dx$. In this case, $\Lambda = \mathbb{Z}$ is a spectrum for $\chi_{[0,1]}dx$. In addition, given a $\mathcal{N}$, we see that, for any $\mathcal{D}$, the corresponding Moran measure has the same spectrum.

2. Proof of Theorem

The mask function of a finite set D in $\mathbb{R}$ is defined by

$$m_D(\xi) = \frac{1}{\#D}\sum_{d\in D} e^{-2\pi i d\xi}.$$

As usual, the Fourier transform of a probability measure μ in $\mathbb{R}$ is defined by