Doctopic: Partial Differential Equations ARTICLE IN PRESS YJMAA:21159

J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A reaction–diffusion model of harmful algae and zooplankton in an ecosystem

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A R T I C L E I N F O

Article history:
Received 22 September 2016
Available online xxxx
Submitted by Y. Yamada

Keywords:
Chemostat
Algal blooms
Inhibition
Global attractor
Weak repellers
Coexistence

A B S T R A C T

This paper is devoted to the investigation of an unstirred chemostat system modeling the interactions of two essential nutrients (i.e., nitrogen and phosphorus), harmful algae (i.e., P. parvum and cyanobacteria), and a small-bodied zooplankton in an ecosystem. To obtain a weakly repelling property of a compact and invariant set on the boundary, we introduce an associated elliptic eigenvalue problem. It turns out that the model system admits a coexistence steady state and is uniformly persistent provided that the trivial steady state, two semi-trivial steady states and a global attractor on the boundary are all weak repellers.

1. Introduction

Harmful algal blooms (HABs) have been a serious problem in many coastal and inland waters worldwide [4,7]. It was known that the algal species, *Prymnesium parvum* (golden algae), is responsible for fish-killing problem, and results in major economic damage [5]. In a reservoir, P. parvum competes for nitrogen and phosphorus with cyanobacteria, which also excrete allelopathic cyanotoxins that inhibit the growth of P. parvum. A small-bodied zooplankton population consume both types of algae for growth, but the dissolved toxins produced by P. parvum also inhibit zooplankton ingestion, growth and reproduction. In order to understand such complex interactions and reactions in an ecosystem, the authors in [6] pro-

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[1] Research supported in part by Ministry of Science and Technology, Taiwan.
[2] Research supported in part by Ministry of Science and Technology, Taiwan; and National Center for Theoretical Sciences (NCTS), National Taiwan University; and Chang Gung Memorial Hospital under grants BMRPD18 and NMRPD5F0541.
[3] Research supported in part by the NSERC of Canada.

http://dx.doi.org/10.1016/j.jmaa.2017.02.034
0022-247X/

Please cite this article in press as: S.-B. Hsu et al., A reaction–diffusion model of harmful algae and zooplankton in an ecosystem, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2017.02.034

posed a well-mixed chemostat system to explore the dynamics of nutrients, P. parvum, toxin(s) produced by P. parvum, cyanobacteria, cyanotoxin(s) produced by cyanobacteria, and zooplankton.

A natural approach to the spatial heterogeneity is to use "unstirred" chemostat, where we will remove the assumption that interactions of nutrients and species proceeds in a well-mixed, spatially uniform habitat. The unstirred chemostat can be regarded as a spatially distributed habitat in which inflow of nutrients occurs at one point and outflow at another, with diffusive transport of nutrients and organisms between these points [3,14]. For simplicity, we ignore the equations of toxins proposed in [6] and assume that inhibitory effects are directly determined by the densities of harmful algae. Accordingly, we modify the model in [6] to obtain the following unstirred chemostat model:

$$\begin{cases} \frac{\partial R}{\partial t} = d\frac{\partial^2 R}{\partial x^2} - q_{1r}f_1(R,S)u_1e^{-\alpha u_2} - q_{2r}f_2(R,S)u_2, & x\in(0,1),\ t>0,\\ \frac{\partial S}{\partial t} = d\frac{\partial^2 S}{\partial x^2} - q_{1s}f_1(R,S)u_1e^{-\alpha u_2} - q_{2s}f_2(R,S)u_2, & x\in(0,1),\ t>0,\\ \frac{\partial u_1}{\partial t} = d\frac{\partial^2 u_1}{\partial x^2} + f_1(R,S)u_1e^{-\alpha u_2} - q_1g_1(u_1)Z, & x\in(0,1),\ t>0,\\ \frac{\partial u_2}{\partial t} = d\frac{\partial^2 u_2}{\partial x^2} + f_2(R,S)u_2 - q_2g_2(u_2)Z, & x\in(0,1),\ t>0,\\ \frac{\partial Z}{\partial t} = d\frac{\partial^2 Z}{\partial x^2} + G(u_1,u_2)Z, & x\in(0,1),\ t>0, \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} \frac{\partial R}{\partial x}(0,t) = -R^{(0)},\ \frac{\partial R}{\partial x}(1,t)+\gamma R(1,t)=0,\ t>0,\\ \frac{\partial S}{\partial x}(0,t) = -S^{(0)},\ \frac{\partial S}{\partial x}(1,t)+\gamma S(1,t)=0,\ t>0,\\ \frac{\partial u_i}{\partial x}(0,t) = \frac{\partial u_i}{\partial x}(1,t)+\gamma u_i(1,t)=0,\ t>0,\ i=1,2,\\ \frac{\partial Z}{\partial x}(0,t) = \frac{\partial Z}{\partial x}(1,t)+\gamma Z(1,t)=0,\ t>0, \end{cases} \tag{1.2}$$

and initial conditions

$$\begin{cases} R(x,0)=R^0(x)\ge 0, S(x,0)=S^0(x)\ge 0, & x\in(0,1),\\ u_i(x,0)=u_i^0(x)\ge 0,\ Z(x,0)=Z^0(x)\ge 0, & x\in(0,1),\ i=1,2. \end{cases} \tag{1.3}$$

Here $R(x,t)$ and $S(x,t)$ denote the complementary nutrient (nitrogen and phosphorus) concentrations at position x and time t; $u_1(x,t)$ and $u_2(x,t)$ represent the densities of P. parvum (golden algae) and cyanobacteria, respectively; $Z(x,t)$ represents the density of small-bodied zooplankton population. $R^{(0)}$ and $S^{(0)}$ are input concentration of nutrients; q_{ir} and q_{is}, $i=1,2$, are the constant nutrient quotas; q_i, $i=1,2$, is the constant algal quota; the constant γ in (1.2) represents the washout constant. We also assume that nutrients and algal species have the same diffusion coefficient d. The term $e^{-\alpha u_2}$ describes the inhibitory effect on $u_1(x,t)$ from $u_2(x,t)$. The response function are given by $f_i(R,S)=\min\{h_{ir}(R),h_{is}(S)\}$, $i=1,2$. The nonlinear functions $h_{ir}(R)$ ($h_{is}(S)$) describe the nutrient uptake and growth rates of species i when only nutrient R (S) is limiting. We assume that the functions $h_{ir}(R)$ and $h_{is}(S)$ satisfy

$$h_{ir}(0)=0,\ h_{ir}'(R)>0\ \forall\ R>0,\ h_{ir}\in C^2, i=1,2.$$

An usual example is the Monod function

$$h_{ir}(R)=\frac{m_{\mathrm{ir}}R}{K_{ir}+R},\ h_{is}(S)=\frac{m_{\mathrm{is}}S}{K_{is}+S}.$$

Both types of algae are consumed by zooplankton, and consumption of the algae supports the growth of the zooplankton. Further, P. parvum $u_1(x,t)$ also inhibits the growth of zooplankton. The function $g_1(u_1)$ represents the relationship between zooplankton and P. parvum, which simultaneously include positive and